# On the optimality of wavefront reconstructors from the gradients at the ELT scale

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# ★ The problem

- "Static" wavefront (WF) reconstruction problem from its local gradients measured in (pseudo) open-loop
  - Shack-Hartmann or pyramid with modulation
- The measurement model (linear)

$$\boldsymbol{d} = \mathbf{S}\boldsymbol{w} + \boldsymbol{e}$$

(data: $\boldsymbol{d}$; sensors: $\mathbf{S}$; layers: $\boldsymbol{w}$; noise: $\boldsymbol{e}$)

$$\begin{bmatrix} \boldsymbol{w} \\ \boldsymbol{e} \end{bmatrix} \sim \mathcal{N}\left( \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \end{bmatrix}, \begin{bmatrix} \boldsymbol{\Sigma}_{ww} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{ee} \end{bmatrix} \right),$$

(zero-mean; turbulence covariance: $\boldsymbol{\Sigma}_{ww}$; noise covariance: $\boldsymbol{\Sigma}_{ee}$)

- Linear WF reconstruction $\hat{\boldsymbol{w}} = \boldsymbol{R}\boldsymbol{d}$ (reconstructor: $\boldsymbol{R}$)
- Criterion : Minimize the mean square error (MSE)

  $$MSE_w(\boldsymbol{R}) := \mathbf{E}\, \|\boldsymbol{R}\boldsymbol{d} - \boldsymbol{w}\|_2^2$$

  - Maximizes the Strehl ratio if the correction applies this WF estimate
  - Min. for **minimum-variance reconstruction**

# ★ Minimum-variance reconstructor

$$\mathbf{R}_{\mathrm{MV}} = \boldsymbol{\Sigma}_{wd}\boldsymbol{\Sigma}_{dd}^{-1}$$
$$= \boldsymbol{\Sigma}_{ww}\boldsymbol{S}^t(\boldsymbol{S}\boldsymbol{\Sigma}_{ww}\boldsymbol{S}^t + \boldsymbol{\Sigma}_{ee})^{-1}$$
$$= (\boldsymbol{S}^t\boldsymbol{\Sigma}_{ee}^{-1}\boldsymbol{S} + \boldsymbol{\Sigma}_{ww}^{-1})^{-1}\boldsymbol{S}^t\boldsymbol{\Sigma}_{ee}^{-1}$$

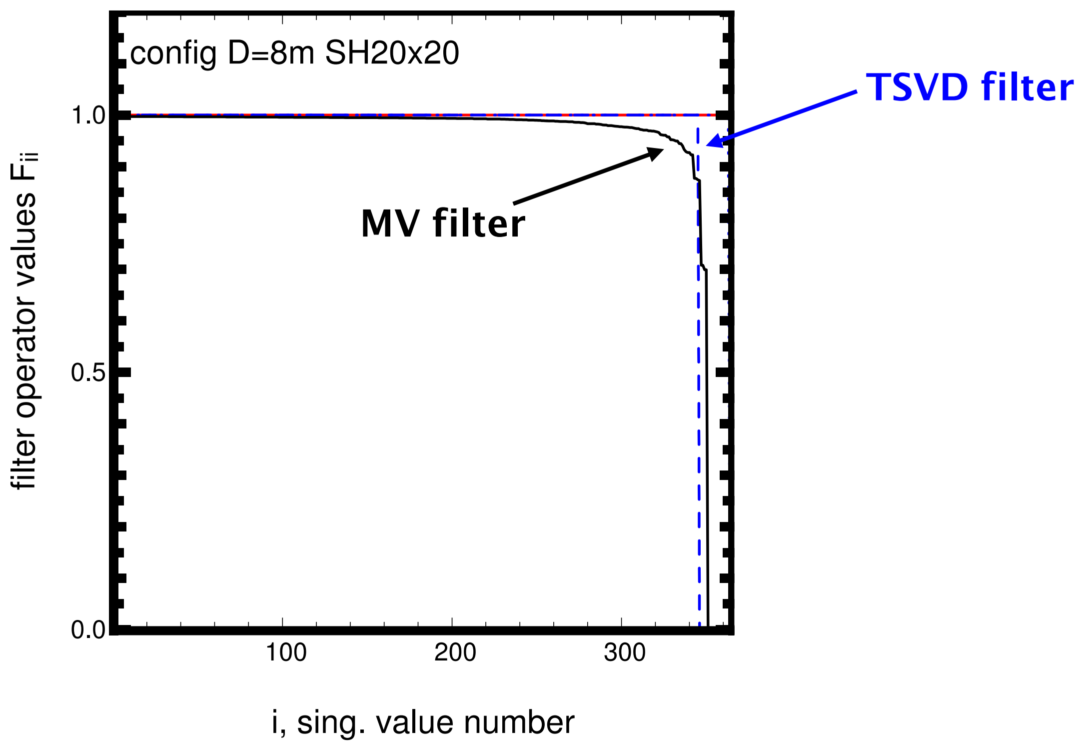
- Since back in the 80's, MV reconstruction for AO is suggested
  - Wallner 82 (SPIE), Wallner (JOSA) 83, Welsh & Gardner 89 (JOSA), Roggeman 92 (CEE), Ellerbroek 94 (JOSA A.), Fusco et al. 2001 (JOSA A), Gilles 2005 (Appl.Opt.), …
  - … and 10 proceedings on it just in AO4ELT4 conference!
- But can still sound like theory or a "dream" in AO since no existing AO systems offered to astronomers includes it (Wrong?)
  - Special mention to "RAVEN **demonstrator**" with MV WF reconstruction!

# Hindrances to MV implementation on AO systems?

1. Scepticism about using « priors » .?
   – Are they really well known?

**The problem is singular so it must be regularized anyway!**

**Example of filter on spectrum when working in Karhunen-Loeve modes**



# Hindrances to MV implementation on AO systems?

1. Scepticism about using « priors » .?
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**The problem is singular so it must be regular … ay!**

2. Computational load issues at the dimensi …
   - Computation of the Matrix inver … )
   - Avoiding inversion with it … this meet latency requirements?
   - No issue at t … es, but the RTC designs of existing AO were n …

**… LTAO, E-ELT HARMONI LTAO … sible at the scale of their ELT systems**

**Most important reason is probably the required time to convince that it is the right way to go (15 to 30 years)!**

3. MV requ … (pseudo) open-loop data!
   - Von Karman statistics do not match the closed-loop residual WF statistics
   - Requires 2 matrix-vector multiply (MVM), not included in current RTCs

**The other MVM involves a very sparse matrix made in advance to the new data arrival. Architecture changes but there should not be any latency issue.**

## Other reconstructors are currently used instead…

- MV reconstructor

$$\mathbf{R}_{\mathrm{MV}} = (\mathbf{S}^t \boldsymbol{\Sigma}_{ee}^{-1} \mathbf{S} + \boldsymbol{\Sigma}_{ww}^{-1})^{-1} \mathbf{S}^t \boldsymbol{\Sigma}_{ee}^{-1}$$

$$\mathbf{K}\mathbf{K}^t = \boldsymbol{\Sigma}_{ww}$$

**Change of basis to rescaled Karhunen-Loève modes**

# Other reconstructors are currently used instead…

- MV reconstructor

$$\mathbf{R}_{\mathrm{MV}} = \mathbf{K}(\mathbf{K}^t\mathbf{S}^t\mathbf{\Sigma_{ee}}^{-1}\mathbf{S}\mathbf{K} + \mathbf{I})^{-1}\mathbf{K}^t\mathbf{S}^t\mathbf{\Sigma_{ee}}^{-1},$$

$$\mathbf{K}\mathbf{K}^t = \mathbf{\Sigma}_{ww}$$

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**RAVEN**
**Planned for:**
**TMT NFIRAOS**
**EELT HARMONI**
**GMT LTAO**

- Tikhonov of zeroth-order reconstructor

$$\mathbf{R}^{\mu}_{\mathrm{Tik}} = \mathbf{K}(\mathbf{K}^t\mathbf{S}^t\mathbf{\Sigma_{ee}}^{-1}\mathbf{S}\mathbf{K} + \mu\mathbf{I})^{-1}\mathbf{K}^t\mathbf{S}^t\mathbf{\Sigma_{ee}}^{-1},$$

**GeMS**

**But works on closed-loop data!**

- Truncated Singular Value Decomposition (TSVD) reconstructor

$$\mathbf{R}^{k}_{\mathrm{TSVD}} = \mathbf{K}(\mathbf{K}^t\mathbf{S}^t\mathbf{\Sigma_{ee}}^{-1}\mathbf{S}\mathbf{K})^{k,\dagger}\mathbf{K}^t\mathbf{S}^t\mathbf{\Sigma_{ee}}^{-1},$$

**SAXO** **LBT AO**
**AOF GLAO** **PKIST**

**But works on closed-loop data!**

$\Rightarrow$ But MSE of any reconstructor is larger than MSE($R_{MV}$)

**Note that among open-loop AO systems, CANARY uses none of those.**

# Interest to study these other reconstructors?

- Showstopper on existing AO systems :
  - Require 2 MVMs to make pseudo-open-loop data

  => require new RTCs

  => But remind that the additional sparse MVM is not on the low-latency path

- Possible upgrade of existing AO systems to include pseudo open-loop data computation ?
  - Then GeMS becomes ~ directly MV reconstruction
  - For AOF, SAXO, PKIST, LBT FLAO, then we would directly know it would not be optimal

- Why would we still choose TSVD instead of Tikhonov if not optimal?
  - Robustness?
  - To noise? To model errors?

=> Analytical study of behavior of the reconstructors w.r.t to model errors

# Introduction of the "reduced model"

- Generalized SVD allows diagonalization of the model

$$\boldsymbol{d} = \mathbf{S}\boldsymbol{w} + \boldsymbol{e} \quad \Leftrightarrow \quad \boldsymbol{v} = \boldsymbol{\Lambda}\boldsymbol{u} + \boldsymbol{\eta}$$

$$\boxed{\boldsymbol{\Sigma}_{ee}^{-1/2}\boldsymbol{S}\mathbf{K} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^t}$$

$$\boldsymbol{v} = \boldsymbol{U}^t\boldsymbol{\Sigma}_{ee}^{-1/2}\boldsymbol{d}$$

$$\boldsymbol{u} = \boldsymbol{V}^t\mathbf{K}^{-1}\boldsymbol{w}$$

$$\boldsymbol{\eta} = \boldsymbol{U}^t\boldsymbol{\Sigma}_{ee}^{-1/2}\boldsymbol{e}$$

$$\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{\eta}\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\boldsymbol{I}_n & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_m\end{bmatrix}\right),$$

# ★ Introduction of the "reduced model"

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$$\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{\eta}\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\boldsymbol{I}_n & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_m\end{bmatrix}\right),$$

$\Rightarrow$ All diagonal reconstructors : $\hat{\boldsymbol{u}} = \mathbf{M}\hat{\boldsymbol{v}}$

$$\mathbf{M}_{\mathrm{MV}} = (\boldsymbol{\Lambda}^t\boldsymbol{\Lambda} + \mathbf{I})^{-1}\boldsymbol{\Lambda}^t$$

$$\mathbf{M}_{\mathrm{Tik}} = (\boldsymbol{\Lambda}^t\boldsymbol{\Lambda} + \mu\mathbf{I})^{-1}\boldsymbol{\Lambda}^t$$

$$\mathbf{M}_{\mathrm{TSVD}} = (\boldsymbol{\Lambda}^t\boldsymbol{\Lambda})^{k,\dagger}\boldsymbol{\Lambda}^t$$

# The noise and the signal in the "reduced model"

- Generalized SVD allows diagonalization of the model

$$\boldsymbol{d} = \mathbf{S}\boldsymbol{w} + \boldsymbol{e} \quad \Leftrightarrow \quad \boldsymbol{v} = \boldsymbol{\Lambda}\boldsymbol{u} + \boldsymbol{\eta}$$

$$\boxed{\boldsymbol{\Sigma}_{ee}^{-1/2}\boldsymbol{S}\mathbf{K} = \boldsymbol{U}\boldsymbol{\Lambda}\boldsymbol{V}^t}$$

$$\boldsymbol{v} = \boldsymbol{U}^t\boldsymbol{\Sigma}_{ee}^{-1/2}\boldsymbol{d}$$

$$\boldsymbol{u} = \boldsymbol{V}^t\mathbf{K}^{-1}\boldsymbol{w}$$

$$\boldsymbol{\eta} = \boldsymbol{U}^t\boldsymbol{\Sigma}_{ee}^{-1/2}\boldsymbol{e}$$

$$\begin{bmatrix}\boldsymbol{u}\\ \boldsymbol{\eta}\end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix}\mathbf{0}\\ \mathbf{0}\end{bmatrix}, \begin{bmatrix}\boldsymbol{I}_n & \mathbf{0}\\ \mathbf{0} & \boldsymbol{I}_m\end{bmatrix}\right),$$

- $\Lambda$ contains the ratio of the signal singular values and the variance of the noise
  - Singular values $\lambda_i < 1$ when the noise dominates

- Karhunen-Loève decomposition of the WF is included but they are re-organized on the singular modes of the operator S

# ★ From the reduced model to the AO calibration

- In practice, one needs a model estimate
  - Analytic, synthetic
  - Pseudo synthetic
  - Measured

- Typical estimates needed :
  - Noise covariance $\widehat{\mathbf{\Sigma}_{ee}}$
  - Interaction matrix $\hat{\mathbf{S}}$ or $\widehat{\mathbf{SK}}$
  - Karhunen-Loève decomposition matrix $\hat{\mathbf{K}}$

- From which, G-SVD provides $\hat{\mathbf{U}}, \hat{\mathbf{\Lambda}}, \hat{\mathbf{V}}$ estimates of $\mathbf{U}, \mathbf{\Lambda}, \mathbf{V}$

- Asumptions : order of estimated singular vectors and values is preserved, s.t.

$$\begin{cases} \hat{\boldsymbol{v}}_i^t \boldsymbol{v}_i > 0, \quad \hat{\boldsymbol{u}}_i^t \boldsymbol{u}_i > 0, \\ \cos(\theta^V_{i,j}) = \hat{\boldsymbol{v}}_i^t \boldsymbol{v}_j \quad \text{and} \quad \cos(\theta^U_{i,j}) = \hat{\boldsymbol{u}}_i^t \boldsymbol{u}_j \end{cases}$$

# ★ From the reduced model to the MSE

- Mean Square Error of a WF reconstructor bounded by

$$
\begin{aligned}
MSE_w(\mathbf{R}) = \mathbf{E}\,\|\hat{\boldsymbol{w}} - \boldsymbol{w}\|_2^2 \leq\ & k_1^2 MSE_U(\mathbf{M}_{\mathrm{MV}}) && \leftarrow \textbf{min. error for perfect knowledge of U, L, V} \\
+\ & 2k_1^2\,\mathbf{trace}((\boldsymbol{I} - \hat{\boldsymbol{U}}^t\boldsymbol{U})\boldsymbol{\Sigma}_{u\hat{u}}) && \leftarrow \textbf{dep. on } \hat{\mathbf{U}} \textbf{ accuracy} \\
+\ & \frac{k_1^2}{2}\|\boldsymbol{D} - \boldsymbol{M}_{\mathrm{MV}}\|_F^2 && \leftarrow \textbf{diff. from the optimal diagonal} \\
+\ & \frac{k_1^2}{2}\|\boldsymbol{N}\|_F^2 && \leftarrow \textbf{off-diagonal terms}
\end{aligned}
$$

with $\boldsymbol{M} = \boldsymbol{D} + \boldsymbol{N}$ ($\boldsymbol{D}$: **diagonal**, $\boldsymbol{N}$: **off-diagonal**) and $k_1$=largest sing. val. of $\mathbf{K}$

⇒ One could a priori reduce each of these 4 terms independently

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# Zeroth-order Tikhonov reconstructor

$$
\begin{aligned}
MSE_w(\mathbf{R}) = \mathbf{E}\,\|\hat{\boldsymbol{w}} - \boldsymbol{w}\|_2^2 \leq\ & k_1^2 MSE_U(\mathbf{M}_{\mathrm{MV}}) \\
&+\ 2k_1^2\,\mathbf{trace}((\boldsymbol{I} - \hat{\boldsymbol{U}}^t \boldsymbol{U})\boldsymbol{\Sigma}_{u\hat{u}}) \\
&+\ \frac{k_1^2}{2}\|\boldsymbol{D} - \boldsymbol{M}_{\mathrm{MV}}\|_F^2 \\
&+\ \frac{k_1^2}{2}\|\boldsymbol{N}\|_F^2
\end{aligned}
$$

# ★ Zeroth-order Tikhonov reconstructor

$$MSE_w(\mathbf{R}) = \mathbf{E}\,\|\hat{\boldsymbol{w}} - \boldsymbol{w}\|_2^2 \leq k_1^2 MSE_U(\mathbf{M}_{\mathrm{MV}})$$

$$+ \ 2k_1^2\, \mathbf{trace}((\boldsymbol{I} - \hat{\boldsymbol{\Lambda}}^t \ldots$$

$$+ \ \frac{k_1^2}{2}\|\boldsymbol{D} \ldots$$

$$\ldots \frac{\ldots}{2} \ldots \sum_i \frac{1}{\hat{\lambda}_i^2 + \mu}(\max_{i \neq j}(\cos(\theta^V_{i,j}))^2$$

Flexibility to reduce the MSE w.r.t. model errors

Selection of μ and $\lambda_i$ allow robustness for wide range of approx. of V and Λ

- R… $\ldots \approx \lambda_i \cos(\theta^V_{i,i}),$

$\ldots \lambda_i \cos(\theta^V_{i,i}) \approx \mu\lambda_i.$

…estimate of V, more the true SV should be underestimated

Do not overestimate large SV. Keep $\hat{\lambda}_i \in [0; \lambda_i + \frac{1}{\lambda_i}]$

$\Rightarrow$ Choose $\mu \approx \min_i \cos(\theta^V_{i,i})^2$

# TSVD reconstructors

$$
\begin{aligned}
MSE_w(\mathbf{R}) = \mathbf{E}\,\|\hat{\boldsymbol{w}} - \boldsymbol{w}\|_2^2 \leq k_1^2 MSE_U(\mathbf{M}_{\mathrm{MV}}) \\
&+ \; 2k_1^2\,\mathbf{trace}((\boldsymbol{I} - \hat{\boldsymbol{U}}^t \boldsymbol{U})\boldsymbol{\Sigma}_{u\hat{u}}) \\
&+ \; \frac{k_1^2}{2}\|\boldsymbol{D} - \boldsymbol{M}_{\mathrm{MV}}\|_F^2 \\
&+ \; \frac{k_1^2}{2}\|\boldsymbol{N}\|_F^2
\end{aligned}
$$

# ★ TSVD reconstructors

$$MSE_w(\mathbf{R}) = \mathbf{E}\,\|\hat{\boldsymbol{w}} - \boldsymbol{w}\|_2^2 \le k_1^2 MSE_U(\mathbf{M}_{\mathrm{MV}})$$

$$+\ 2k_1^2\,\mathbf{trace}((\boldsymbol{I} - \hat{\boldsymbol{U}}^t \boldsymbol{U} \dots$$

$$+\ \frac{k_1^2}{2}\|\boldsymbol{D} - \dots \frac{\dots(\theta_{i,i})}{\hat{\lambda}_i}\left(\lambda_i + \frac{1}{\lambda_i}\right)\Big]^2 + \frac{1}{8}(n-k)$$

$$+\ \frac{1}{2\hat{\lambda}_{k+1}^2}(n-k)^2\left(\max_{i \neq j}\ \cos(\theta_{i,j}^V)\right)^2$$

**TSVD: not as much flexibility as Tikhonov to reduce MSE**

- Red… $(\lambda_i + \frac{1}{\lambda_i}) \cos\theta_{i,i}^V$

…guideline for the truncation choice k

⇒ Require to optimize the balance between diagonal and off-diagonal errors

# Preliminary results and further work

- Analytical study results: greater flexibility of Tikhonov reconstructor on TSVD to reduce the MSE w.r.t model estimate errors
  - However many existing AO systems work on TSVD reconstructors
  - Next step : numerical quantification and comparison using AO simulations

- Started review of ”calibration” methods applied on existing AO
  - So many different ways to estimate the model (pseudo-synthetic, modal, ...)
  - Lead to different structure of the errors on the operators
  - Good framework to study propagation of these errors toward the SV and singular vectors estimates, as well as toward the MSE

- Flexibility of Tikhonov reconstructor w.r.t. model errors will directly extend its benefits to the MV reconstructors planned on future ELT AO systems, since it is just a particular case of Tikhonov
  - ... And on some existing AO systems if RTCs are upgraded to 2 MVMs!

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*For the details on what is implemented in some existing AO systems*

# Hindrances to MV implementation on AO systems?

- Scepticism about using « priors » ?
  - Are they really well known?

- How to compute such matrix-vector multiplication (MVM) at the dimensions of the ELTs?
  - Full dense MVM because of
  - Matrix inversion (may need to be often updated)
  - Iterative methods are smart but can they meet latency requirements?
  - Unexpected issues at the current 8-10m telescopes scales, but RTCs designed were made ~ decade ago

- Tomographics reconstruction dimensions increase with number of layers and FOV size

- Most systems use closed-loop data!
  - Von Karman statistics do not match the closed-loop residual WF statistics