Dynamics of the Milky Way

        II. Fundamentals of Stellar Dynamics

Part 2:  Stellar Orbits and Jeans’ Theorem

Part 3:  The Galactic Bulge and Bar

Part 4:  The Galactic Disk and Halo
Collisionless Stellar Dynamics

\[
\begin{align*}
\frac{df}{dt} &= \frac{2f}{\partial t} + v \cdot \frac{2f}{\partial x} - \frac{\partial \phi}{\partial x} \cdot \frac{2f}{\partial y} = 0 \\
\bigtriangledown^2 \phi &= 4\pi G \int f \, d^3v \\
f \text{ contains all information} \\
\text{streaming velocity } u(x) \\
\sigma_{ij} &= \int d^3v \, (v_i - u_i)(v_j - u_j) f \\
\text{velocity dispersion tensor } \sigma(x) \\
\end{align*}
\]
Moments of CBE – Stellar Hydrodynamics

\[ \int d^3 v \frac{df}{dt} = 0 \Rightarrow (\frac{\partial}{\partial t} + u \cdot \nabla) s = -L \cdot u \]

1st (momentum or Jeans’ eq)

\[ \int d^3 v \nabla \cdot \frac{df}{dt} = 0 \Rightarrow (\frac{\partial}{\partial t} + u \cdot \nabla) u = -\frac{\partial \Phi}{\partial x} - \frac{1}{s} \Gamma \frac{\partial s}{\partial x} \]

- hierarchy not closed - no eqn. of state
- stellar fluid compressible \& anisotropic
- dist \( f \) not guaranteed for given solution
Spherically Symmetric Jeans Equations

Assume both potential and kinematics spherically symmetric
\[ \implies \text{no streaming motions, all mixed second-order moments vanish, } v_k^2 = \sigma_k^2 = \sigma_{kk}^2, k = (r, \theta, \phi), \sigma_\theta^2 = \sigma_\phi^2 \]
\[ \implies \text{only one non-trivial Jeans equation} \]
\[ \frac{d\rho \sigma_r^2}{dr} + \frac{\rho}{r} \left( 2\sigma_r^2 - \sigma_\theta^2 - \sigma_\phi^2 \right) + \rho \frac{d\Phi}{dr} = 0, \]

Notice: spherical Jeans equation not sufficient to determine the dynamics: if \( \rho(r), \Phi(r) \) presumed known, remain two unknown functions \( \sigma_r \) and \( \sigma_\theta \), but only one equation. Contrary to hydrostatic equilibrium eq. for fluid where pressure is determined from density and potential, or \( \Phi(r) \) from pressure and density of confined gas.

If velocity dispersion tensor isotropic, \( \sigma_{kl}^2 = \sigma^2 \delta_{kl} \), \( \beta = 0 \implies \]
\[ \rho \sigma^2 = \int_r^\infty \rho \frac{d\Phi}{dr} dr. \]
If velocity dispersion tensor \( \text{isotropic} \), \( \sigma_{kl}^2 = \sigma^2 \delta_{kl}, \beta = 0 \implies \)
\[
\rho \sigma^2 = \int_r^\infty \rho \frac{d\Phi}{dr} \, dr.
\]

In general, define anisotropy parameter
\[
\beta = 1 - \frac{\sigma_\theta^2}{\sigma_r^2}
\]
and substitute \( d\Phi/\, dr = GM(r)/r^2 \implies \)
\[
M(r) = -\frac{r \sigma_r^2}{G} \left( \frac{d \ln \rho}{d \ln r} + \frac{d \ln \sigma_r^2}{d \ln r} + 2\beta \right).
\]

This equation valid for distribution of luminous stars \( \rho(r) \) in any (self-consistent or external) potential \( \Phi \), including dark matter contribution
\[\implies \text{used for estimating total mass: additional assumption on} \beta(r) \text{ required (Binney & Mamon 1982), then get} \sigma_r(r) \text{ from measured projected velocity dispersion and insert.} \]
Stellar hydrodynamic Pressure

Hydrostatic Eq. for gas sphere:

\[ s \frac{GM(\nu)}{r^2} = - \frac{dp}{dr} \]

Spherical Jeans Eq. for 'static' stellar system:

\[ s \frac{GM(\nu)}{r^2} = - \frac{d}{dr}(s \sigma^2) - \frac{2 \rho \sigma^2}{r} \]

Kinematic pressure gradient

\[ \frac{d}{d\nu}(s \sigma^2) \]

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"force"

Anisotropy

Because stellar system is collisionless, such pressure can be anisotropic (no scattering).
For axisymmetric system in steady state, three second-order Jeans equations:

\[
\frac{\partial (\rho \overline{v_R^2})}{\partial R} + \frac{\partial (\rho \overline{v_R v_z})}{\partial z} + \rho \left( \frac{\overline{v_R^2} - \overline{v_\phi^2}}{R} + \frac{\partial \Phi}{\partial R} \right) = 0,
\]

\[
\frac{\partial (\rho \overline{v_R v_\phi})}{\partial R} + \frac{\partial (\rho \overline{v_\phi v_z})}{\partial z} + \frac{2 \rho \overline{v_R v_\phi}}{R} = 0,
\]

\[
\frac{\partial (\rho \overline{v_R v_z})}{\partial R} + \frac{\partial (\rho \overline{v_z^2})}{\partial z} + \frac{\rho \overline{v_R v_z}}{R} + \rho \frac{\partial \Phi}{\partial z} = 0.
\]

These equations connect six velocity moments \(\rightarrow\) dynamics underconstrained. If only azimuthal streaming motions, \(\overline{v_\phi}\), and \(\sigma_{ij} \text{ meridionally isotropic}\), i.e., \(\sigma_R^2 = \sigma_z^2 = \sigma^2\), then all mixed moments vanish and the first and third eqs. become

\[
\frac{\partial (\rho \sigma^2)}{\partial R} + \frac{\rho (\sigma^2 - \overline{v_\phi^2})}{R} + \rho \frac{\partial \Phi}{\partial R} = 0,
\]

\[
\frac{\partial (\rho \sigma^2)}{\partial z} + \rho \frac{\partial \Phi}{\partial z} = 0.
\]

Dynamics of the Milky Way — Part 1: Introduction and Fundamental Stellar Dynamics
Meridionally Isotropic Case

\[ \frac{\partial (\rho \sigma^2)}{\partial R} + \frac{\rho (\sigma^2 - \bar{v}_{\phi}^2)}{R} + \rho \frac{\partial \Phi}{\partial R} = 0, \]

\[ \frac{\partial (\rho \sigma^2)}{\partial z} + \rho \frac{\partial \Phi}{\partial z} = 0. \]

For \( \rho(R, z) \) and hence potential \( \Phi(R, z) \) known, solve second of these eqs. for the meridional velocity dispersion

\[ \sigma^2(R, z) = \frac{1}{\rho} \int_{z}^{\infty} \rho \frac{\partial \Phi}{\partial z} \, dz, \]

and the first equation then gives the mean-square azimuthal velocity, \( \bar{v}_{\phi}^2 = \sigma_{\phi}^2 + \bar{v}_{\phi}^2 \):

\[ \bar{v}_{\phi}^2(R, z) - \sigma^2 = R \frac{\partial \Phi}{\partial R} + \frac{R}{\rho} \frac{\partial}{\partial R} \int_{z}^{\infty} \rho \frac{\partial \Phi}{\partial z} \, dz. \]

Notice: \( \bar{v}_{\phi}^2 \) fixed, but streaming \( \bar{v}_{\phi} \) velocity undetermined by the Jeans equations (Satoh 1980, Binney, Davies & Illingworth 1990).
**JAM-Models**

Velocity ellipsoids (VE) for ETGs from Schwarzschild models often as in figure. Therefore consider Jeans models with cylindrical VE such that \( v_R v_z = 0 \) and \( v_R^2 = bv_z^2 \). Jeans eqs become

\[
\frac{b v_R^2 - v_y^2}{R} + \frac{\partial (b v_y v_z)}{\partial R} = -\nu \frac{\partial \Phi}{\partial R}
\]

\[
\frac{\partial (v_y v_z)}{\partial z} = -\nu \frac{\partial \Phi}{\partial z},
\]

with solution

\[
v_y^2(R, z) = \int_z^\infty \nu \frac{\partial \Phi}{\partial z} dz
\]

\[
v_y^2(R, z) = b \left[ R \frac{\partial (v_y v_z)}{\partial R} + v_y v_z^2 \right] + R \nu \frac{\partial \Phi}{\partial R}.
\]

Projected \( \sigma \) and \( v \) (a la Satoh ‘80) good fit to projected Sauron data with M/L and anisotropy recovered within \( \sim 0.5 \).

Cappellari ‘08
\[ \int d^3v \int d^3x \ x_j v_k \frac{df}{dt} = 0, \text{ then (9)} \]

\[ \frac{1}{2} \ddot{I}_{jk} = 2T_{jk} + \Pi_{jk} + W_{jk} \]

**moment of inertia**
\[ I_{jk} = \int d^3x \ x_j x_k \rho(x) \]

**streaming kE**
\[ T_{jk} = \frac{1}{2} \int d^3x \ u_j u_k \rho(x) \]

**pressure**
\[ \Pi_{jk} = \int d^3x \ \langle (v_j - u_j)(v_k - u_k) \rangle \rho(x) \]

**potential**
\[ W_{jk} = -\int d^3x \ x_j \frac{\partial \phi}{\partial x_k} \rho(x) \]

**Note:** a global equiv., e.g. \[ \Pi_{jk} = \int \sigma_{jk} d^3x \]

- applies to system as a whole
- Jeans' eqns. at \( \nabla \times \)
- surface terms \( S_{jk} \) when applied to part of system
Example: Flattening of Oblate Spheroids

Tensor Virial Theorem for axisymmetric system, $z$-axis = symmetry axis

\[ \frac{d^2I}{dt^2} = 0 \]

\[
\begin{cases}
W_{xx} = W_{yy} & W_{ij} = 0 & i \neq j \\
T_{xx} = T_{yy} & T_{ij} = 0 & i \neq j \\
\Pi_{xx} = \Pi_{yy} & \Pi_{ij} = 0 & i \neq j 
\end{cases}
\]

Remaining non-trivial equations:

\[ 2T_{xx} + \Pi_{xx} + W_{xx} = 0 \]
\[ 2T_{zz} + \Pi_{zz} + W_{zz} = 0 \]

or

\[ \frac{2T_{xx} + \Pi_{xx}}{2T_{zz} + \Pi_{zz}} = \frac{W_{xx}}{W_{zz}} \]

For spheroids with axes $(a=b, c)$ the ratio $W_{xx}/W_{zz}$ is independent of the density profiles:

\[ \frac{W_{xx}}{W_{zz}} = f\left(\frac{a}{c}\right) = \left(\frac{a}{c}\right)^{0.89} \]

Hence the ratio of kinetic energies along the different axes is simply related to the axis ratio of the figure.
Rotational flattening , isotropic pressure
\[ T_{xx} = T_{yy} = T_{zz} = M \langle \sigma^2 \rangle \]
\[ T_{zz} = 0, \quad T_{xx} + T_{yy} = \frac{1}{2} M \langle \sigma^2 \rangle \]
then
\[ \frac{\langle \sigma \rangle}{\langle \sigma^2 \rangle^{0.89}} = \sqrt{2} \left( \frac{\sigma}{\langle \sigma \rangle} \right)^{0.89} - 2. \]

Observe typical projected \( \nu \mu \)
projected \( \sigma \) within ~Re.

\( \frac{\nu \mu}{\sigma} \approx \frac{3}{4} \frac{\langle \sigma \rangle}{\langle \sigma^2 \rangle^{0.89}} \)

Fokker's formula (0.89 \rightarrow 1):
\[ \frac{\nu \mu}{\sigma} \approx \frac{\sqrt{3}}{1 - \epsilon} \]

Example: \( \gamma_a = 0.7 \rightarrow \nu \mu/\sigma = 0.8 \)

Flattening by anisotropy only
\[ T_{xx} = \frac{1}{2} \sigma_x^2, \quad T_{zz} = \frac{1}{2} \sigma_z^2 \]
then
\[ \frac{\sigma_z}{\sigma_x} \approx \left( \frac{\sigma}{\langle \sigma \rangle} \right)^{0.45} \]

Example: \( \gamma_a = 0.7 \rightarrow \sigma_z/\sigma_x = 0.87 \)

\[ \frac{\langle v \rangle^2 + 2 \sigma_x^2}{\sigma_z^2} = 2 \left( \frac{\sigma}{\langle \sigma \rangle} \right)^{0.89} = \frac{2}{(1 - \epsilon)(0.89)} \]

Note: need little anisotropy to flatten non-rotating systems, but a fair amount of rotation.
Flattening of Bulges and Elliptical Galaxies

- Classical bulges rotate faster than elliptical galaxies, \( \sim \) near oblate-isotropic line.
- Pseudobulges (disky bulges and bulges in barred galaxies) rotate faster than classical bulges.

Kormendy ‘11
Dynamics of the Milky Way

Part 2:
Stellar Orbits and Jeans Theorem
Orbits in spherical, axisymmetric, triaxial potentials
Orbits in disks and in rotating potentials
Integrals of motion, action-angles, integrable and non-integrable potentials
Jeans’ Theorem, equilibrium distribution functions for spherical and axisymmetric systems
Self-consistent dynamical models, Schwarzschild’s method
Orbits in Mean-Field Potentials

• 2-body relaxation time $\gg$ age of galaxy, therefore potential perturbations from individual stars can be neglected

• Sufficient to consider orbits in mean-field potential

• This can be bumpy, e.g., infalling dark matter subhalos, spiral arms in the disk, massive molecular cloud complexes, etc.

• First step is to study orbits in typical smooth potentials

• Later, can consider perturbations to these orbits
Orbits in Spherical Potentials

**Circular Velocity**
\[ v_c(r) = \frac{GM(r)}{r} \]

**Escape Velocity**
\[ v_e(r) = -2\phi(r) \]

**Eqs. of Motion**
\[ \ddot{r} - \dot{\phi}^2 = -\frac{\partial \Phi}{\partial r} \]
\[ r^2 \dot{\phi} = L = \text{constant} \]

**Energy Eq.**
\[ E = \frac{1}{2} \dot{r}^2 + \frac{1}{2} (\dot{\phi})^2 + \Phi(r) \]
\[ = \frac{1}{2} \dot{r}^2 + \frac{1}{2} \frac{L^2}{r^2} + \Phi(r) \]

**Orbit eqn**
\[ \frac{d^2\psi}{dt^2} + u = -\frac{\Phi(r)}{L^2/r^2}, \quad u \equiv \frac{\dot{r}^2}{2} + \Phi(r) \]

**Validity regions**
- Planar: \[ \frac{L}{T} < r < \infty \]
- Rosettes: \[ L < \frac{T}{T} < 2 \]

**Limiting Radii**
\[ \frac{dr}{dt} = 0 = \pm \sqrt{2[E - \Phi(r)] - \frac{L^2}{r^2}} \]

**Radial Period**
\[ T_r = 2 \int_{r_0}^{r_e} \frac{dr}{\sqrt{2E - \Phi(r) - \frac{L^2}{r^2}}} \]

**Angle per radial oscillation**
\[ \Delta \phi = 2 \int_{r_0}^{r_e} \frac{L}{r^2} \frac{dr}{\sqrt{2E - \Phi(r) - \frac{L^2}{r^2}}} \]

**Action variables** \[ J_r, J_\theta, J_\phi \] (\( \leftrightarrow \) \( E_l, L_z \))

**Define torus**
- 4th integral: degeneracy
Integrable Potentials

- Spherical $\Phi$ is an example of a separable $\Phi$, i.e. the Hamilton-Jacobi eqn can be separated in suitable coordinates, and the eqs of motion can be integrated by quadratures.
- Separable potentials are a subset of integrable potentials.
- Consider Hamiltonian system with eqs of motion

$$H(x, p) = \frac{1}{2}p^2 + \Phi(x)$$

$$\dot{x} = \frac{\partial H}{\partial p}, \quad \dot{p} = -\frac{\partial H}{\partial x}$$

- $I(x,p)$ is called an integral of motion if it is constant along every orbit. The integral $I(x,p)$ and $H$ are called in involution iff $\{I,H\}=0$, where

$$\frac{dI}{dt} = \{I, H\} \equiv \frac{\partial I}{\partial x} \frac{\partial H}{\partial p} - \frac{\partial I}{\partial p} \frac{\partial H}{\partial x}$$

- Any function of the integrals is itself an integral of motion.
Integrable Potentials: Liouville’s Theorem

A Hamiltonian system with \( n \) degrees of freedom (i.e., \( 2n \)-dim phase-space) which has \( n \) independent integrals in involution is integrable by quadrature.

- The surfaces defined by \( I_1, \ldots, I_n = \text{const} \) are \( n \)-dimensional tori. On the tori, the motion is quasi-periodic, and there exists a system of canonical action-angle coordinates \( (J, \theta) \) in which the eqs of motion can be written as

\[
\begin{align*}
\frac{d\theta}{dt} &= \frac{\partial H}{\partial J} = \Omega(J) \quad \cdot \quad \cdot \\
\frac{dJ}{dt} &= 0; \quad H = H(J)
\end{align*}
\]

The \( \gamma_i \) are the \( n \) independent loops around the torus.
Action-Angle Variables and Quasi-Periodicity

- The $\Omega(J)$ in these equations of motion are the orbital frequencies, which are constant on each torus. All angle variables evolve $2\pi$-periodically and linearly with time; the complete motion is therefore quasi-periodic in time:

$$x(t) = \sum_n x_n \cos(n.\Omega t + \psi_n)$$

- Example: harmonic oscillator in 1D – 1D torus with area $J$

$$H = \frac{1}{2}(p^2 + x^2) = \frac{1}{2}J$$

$$p = J^{1/2} \cos \theta; \quad x = J^{1/2} \sin \theta$$

$$\Omega = \frac{1}{2}; \quad \theta = \theta_0 + \frac{1}{2}t$$
Advantages of Action-Angle Coordinates

- Actions exist also for isolated tori (later)
- Actions have the property of adiabatic invariance
- The angle variables are natural coordinates to label points on the invariant tori
- Best-suited for perturbation analysis in order to treat near-integrable systems
Orbits in Planar Non-Axisymmetric Potentials

• These are no longer integrable – can sometimes be obtained by perturbation theory, but generally require numerical integration. Many have ‘near-integrable’ orbit structure.

• Illustrate properties of more general orbits, and describe motion in meridional plane of axisymmetric potentials and in principal plane of triaxial potentials. Example: logarithmic $\Phi=\ln(x^2+y^2/q^2+c^2)$

• Box – loop orbit structure, illustrated by integrable Staeckel potentials.

• In general, additional resonant orbit families as well as chaotic orbits; these become important when “perturbations” (singular densities, non-elliptical isophotes, large flattening, central black hole) grow large.
Elliptical coordinates $\lambda, \mu$

Separable potential

$$\Phi(\lambda, \mu) = \frac{\vartheta(\lambda) - \vartheta(\mu)}{\lambda - \mu}$$

then there exists an exact second integral $I_2$

Orbits bounded by coordinate surfaces

$$p_\lambda^2 = p_\lambda^2 \left( E, I_2; \beta - \kappa; \lambda \right)$$

$$p_\mu^2 = p_\mu^2 \left( E, I_2; \beta - \kappa; \mu \right)$$

Inside focus $E_c (\mu = -\beta)$: ↔ 2D harmonic oscillator stable $x, y$-axial orbits

Outside focal energy: box and loop orbits

Dynamics of the Milky Way --- Part 2:
Stellar Orbits and Jeans Theorem
Box & loop orbits in a cored 2D potential (left)
Resonant orbits in a cuspy potential (right)
Properties of Surface of Section

• Surface-of-section is a tool to study these orbit structures. Here, energy surface $E = H(x, y, x_d, y_d)$ is a 3D surface in phase-space $(x, y, x_d, y_d)$. s.o.s. is a slice through this 3D surface, e.g., \( \{ H(x, y, x_d, y_d) = E; y = 0 \} \).

• Properties of s.o.s:
  – Closed orbits $\Rightarrow$ finite number of points in s.o.s.
  – Non-closed orbits with eff. 2\textsuperscript{nd} integral $\Rightarrow$ smooth curves in s.o.s.
  – Stochastic (chaotic) orbits $\Rightarrow$ area-like distribution of points in s.o.s without clear organisation, limited by invariant curves
  – Area-preserving (Hamiltonian)

• Useful to elucidate the orbit structure of a 2D potential
Theorems from Non-Linear Dynamics

• Almost all Hamiltonians are non-integrable. The set of integrable potentials has measure zero.

• Kolmogorov-Arnold-Moser Theorem for near-integrable $\Phi$s: for $H = H_0 + \varepsilon H_1$ with $H_0$ integrable, most orbits still lie on tori for $\varepsilon \to 0$. $M(\text{tori}) \to 1$ for $\varepsilon \to 0$.

• Poincare-Birkhoff Theorem: orbits first become irregular (tori break up) near resonances of $H_0$.

highly simplified; cf. Arnold, Classical Mechanics
Broken-up Resonant Families and Integrals

- Resonances are dense in phase-space
- Chaotic regions about various resonances may create intricate network
- Integrals of motion as analytic functions I(x,p) do not even exist locally