Dynamics of the Milky Way

         II. Fundamentals of Stellar Dynamics
Part 2:  Stellar Orbits and Jeans’ Theorem
Part 3:  The Galactic Bulge and Bar
Part 4:  The Galactic Disk and Halo
Surface of Section

Box & loop orbits in a cored 2D potential (left)
Resonant orbits in a cuspy potential (right)
Near-Integrable Galactic Potentials

• Many simple galaxy-like potentials turn out to be near-integrable, i.e., a large fraction of phase-space is still occupied by regular orbits (those on invariant tori). (Note that potentials in accreting galaxies may be more complicated.)

• According to the KAM theorem, many orbits in near-integrable potentials are confined to invariant tori. For these orbits one can determine values for action integrals.

• However, resonant tori are dense within the phase-space; there is always one arbitrarily close to any surviving torus. Therefore, no second or third integral can exist as analytic function of phase-space, only the energy remains.

• Despite this, astronomers often speak of “three-integral models” – what they mean is that the models explore the 3D torus-space, and (!?) that most orbits are regular or can be approximated as such.
Orbits in Axisymmetric Potentials

• Axisymmetric $\Phi \rightarrow L_z = \text{const}$; then can define effective potential

\[ \Phi_{\text{eff}} = \Phi + \frac{L_z^2}{2R^2} \]

• Motion reduced to 2D equations in the meridional plane

\[ \ddot{R} = -\frac{\partial \Phi_{\text{eff}}}{\partial R}, \quad \ddot{z} = -\frac{\partial \Phi_{\text{eff}}}{\partial z} \]

• The meridional plane $\phi = \phi(t)$ rotates with the star,

\[ \dot{\phi} = \frac{L_z}{R} \]

• $L_z/2R^2$ creates a centrifugal barrier for most orbits (unless $L_z=0$); orbits confined between minimum and maximum radii
Orbits in \( \Phi_L(R,z) \)

Orbits in \( \Phi_L = v_0^2/2 \left( R^2 + z^2 + R_0^2 \right) \); figures from Binney & Tremaine (1987)

Figure 3-3. Two orbits in the potential of equation (3-50) with \( q = 0.9 \). Both orbits are at energy \( E = -0.8 \) and angular momentum \( L_z = 0.2 \), and we assume \( v_0 = 1 \).

Figure 3-4. Points generated by the orbit of Figure 3-3a in the \((R, \dot{R})\) surface of section. If the total angular momentum \( L \) of the orbit were conserved, the points would fall on the dotted curve. The full curve is the zero-velocity curve at the energy of this orbit.
Characterizing orbits in axisymmetric potential

- Orbits on surface of constant energy are 2-parameter family
- Label them with action variables, approximate integrals, turning points in near-integrable case.
- Example: shape invariants based on $L_z$ and approximate integral, similar to turning points (Dehnen & OG. 93)
- Then, construct $DF(E,I_2,I_3)$. Many exist for the same $\rho(R,z)$.

Dynamics of the Milky Way --- Part 2:
Stellar Orbits and Jeans Theorem

Orbits in Triaxial Potentials

Separable Staeckel potentials (de Zeeuw 1985) illustrative for triaxial potentials with a constant-density core and a steep outer density gradient.

Stable periodic orbits are loops in (x,y) and (y,z) planes, and linear orbit along x-axis.

More general potentials again support resonant and some chaotic orbit families. They can occupy significant parts of phase-space depending on the potential, particularly at large energy.

Schwarzschild (1979) and ff.
Boxlet Orbits in Scale-Free Logarithmic Potentials

\[ \Phi = \ln(x^2 + y^2) \]
Orbits of Disk Stars – Epicycle Approximation

• At given $L_z$, the effective potential $\Phi_{eff} = \Phi + \frac{L_z^2}{2R^2}$ has minimum for circular orbit with $R=R_g$

• Taylor expansion of $\Phi_{eff}$ around $(R_g, z=0)$ will be quadratic in the coordinates $x=R-R_g$, $y=\phi-\phi_g$, $z$, i.e.

$$\Phi_{eff} = \Phi_{eff}(R_g, 0) + \frac{1}{2}\kappa^2 x^2 + \frac{1}{2}\nu^2 z^2$$

• In this approximation, motions will be harmonic

$$R(t) = A \cos(\kappa t + a) + R_g$$
$$z(t) = B \cos(\nu t + b)$$
$$\phi(t) = \Omega_g t + \phi_0 - \frac{2\Omega_g A}{\kappa R_g} \sin(\kappa t + a)$$

and trace out retrograde epicycle
Orbits in Rotating 2D Potentials

• Consider motion in coordinate system rotating with angular frequency $\Omega$. Hamiltonian becomes

$$H_J = H_0 - \Omega \cdot \mathbf{L} = H_0 - \frac{1}{2} \Omega^2 R^2$$

• $H_0$ includes the gravitational potential, so we have effective potential

$$\Phi_{\text{eff}} = \Phi - \frac{1}{2} \Omega^2 R^2$$

• $\Phi_{\text{eff}}$ is time-independent; $H_J$ Jacobi energy is conserved

• $\Phi_{\text{eff}}$ has 5 stationary points, the Lagrange points L1-L5.

Motion around
• L3 always stable (minimum)
• L1,L2 always unstable (saddle points)
• L3,L4 stable or unstable, (maxima of $\Phi_{\text{eff}}$)
Orbits in Weak Rotating Bars

- Consider weak potential perturbation $\Phi_1 = \Phi_b(R) \cos m\phi$ rotating with pattern speed $\Omega_b$. Linearized EOM become

$$\ddot{R}_1 + \kappa_0^2 R_1 = -\left[ \frac{d\Phi_b}{dR} + \frac{2\Omega \Phi_b}{R(\Omega - \Omega_b)} \right]_{R_0} \cos [m(\Omega_0 - \Omega_b)t] + \text{constant},$$

index 0 denotes unperturbed orbit. EOM of harmonic oscillator driven at frequency $m(\Omega_0 - \Omega)$. Using also $t = \phi_0 / (\Omega_0 - \Omega_b)$, solution is

$$R_1(\phi_0) = C_1 \cos \left( \frac{\kappa_0 \phi_0}{\Omega_0 - \Omega_b} + \psi \right) + C_2 \cos (m\phi_0),$$

with

$$C_2 \equiv -\frac{1}{\Delta} \left[ \frac{d\Phi_b}{dR} + \frac{2\Omega \Phi_b}{R(\Omega - \Omega_b)} \right]_{R_0} \quad \Delta \equiv \kappa_0^2 - m^2(\Omega_0 - \Omega_b)^2$$

Note resonant denominators at corotation and Lindblad resonances. Closed orbits ($C_1 = 0$) change shape at resonance.
Lindblad Resonances

![Graph showing Lindblad Resonances](image)

Dynamics of the Milky Way --- Part 2: Stellar Orbits and Jeans Theorem

Periodic Orbits in Barred Potential

Figure from Sellwood & Wilkinson ‘93
Main Orbits in Rotating Barred Potentials

For non-linear barred potential main stable closed orbits are

- prograde long axial x1,
- prograde short axial x2 (in \(E1<EJ<E2\)),
- retrograde x4 (usually unused),
- possibly families around L4, L5,
- prograde x3 family unstable
  (notation due to Contopoulos)

Non-closed (‘epicyclic’) x1-orbits leave hole near origin (Coriolis force) and may develop ‘ansae’.
Chaotic Orbits

- Regular orbit families trapped around periodic orbits (fixed points of s.o.s. map):
  - X1: near centre
  - X2: on the right
  - X4: on the left
- Resonant islands around outer boundaries of the regular families
- Beyond these many chaotic orbits. NB large no. of resonances near corotation

Figure from Sellwood & Wilkinson ‘93
Orbits in 3D Rotating Potentials

Periodic orbits in a rotating Ferrer’s bar, 1:4:10, \( R_b = R_{\text{cor}} \)
- backbone of any orbit distribution
- Families branch off from planar x1 family, in short instability strips
- Notation: \( N(\text{oscill}) \) in \( \mathbf{R} : \phi : z \); symmetric (R) antisymmetric (L) in the \((y,z)\)-plane

Pfenniger ‘84
Orbits in 3D Rotating Potentials

- Periodic orbits in an N-body model
- Symmetric (in yz-plane) 2:1:2s family (dotted), stable near plane
- Anti-symmetric 2:1:2a family (full), stable at higher energies
- Most orbits in this model trapped around planar x1 and 2:1:2a families

Pfenniger & Friedli ‘91
Jeans' Theorem

Assume galaxy = stellar system is in steady state \( \frac{\partial \rho}{\partial t} = 0 \)

\[ f(x, v) = f(I_1, \ldots, I_K) \]

\( I_K \) are the independent integrals of motion of stellar orbits in \( \phi \).

Only isolating integrals (not phase) should be used (Lynden-Bell)

**Classical Integrals**

\( \phi \) spherical axisymmetric general

\( E, L \) \( E, L_2 \) \( E \)

**Non-classical Integrals**

\( \phi \) Stäckel axisymmetric general

\( I_2, I_3 \) 'third' approx. ?
Global Nature of Jeans’ Theorem

⇒ Jeans’ theorem implies that very different parts of the system are dynamically coupled. Can’t use local models except when consisting of only near-circular orbits
Generalized Jeans' Theorem

• For integrable potentials,
  \( f = f \) (analytic isolating integrals)
• For near-integrable potentials,
  \( f = f \) (invariant tori, or actions, or numerical orbits)
• For systems with significant chaotic regions,
  \( f = f \) (invariant ensembles)
  = region of ergodic, fully mixed chaotic orbits

NB: orbits in some chaotic regions may take a long time before they fill the region; such systems may dynamically evolve

NB2: chaotic orbits fill accessible part of energy surface, bounded by potential contours. Rounder than density contours, hence mostly chaotic systems may not be self-consistent