LAW II: THE AREA LAW (1609 and 1621)

1. Modern form
A planet moves in its elliptic orbit so that the line joining the planet to the Sun sweeps out equal areas in equal intervals of time.

2. Kepler stated his result in *Astronomia nova* (1609), in Chapter 60 (KGW III p.377 lines 3-5; Donahue p.593): 'the time [taken by Mars as it moves in its orbit] is measured by the area'.

3. What Kepler initially discovered
The figure below shows the planet at a position P on its path CFD whose (major) diameter is CD, circumscribed by the circle diameter CD centre B. Q is a typical point of that circle, determined by the angle $\angle QBC = \beta$ at the centre, with QPH the ordinate linking the associated points Q and P. A is the position of the Sun, and we follow Kepler by describing the point A as 'the eccentric point' (because it is 'off-centre'). Then for consistency with Law I, we shall express AB, the eccentric distance, as a proportion ae of the radius a of the circle. (For the separation of the circle and the ellipse to be visible, the eccentric distance always has to be greatly exaggerated.)

Kepler established that when the planet is at position P in its elliptic orbit, the time taken is represented by the area of the corresponding sector: in the figure, the time taken to reach P (from starting-point C) is proportional to area *PAC* of the ellipse swept out.

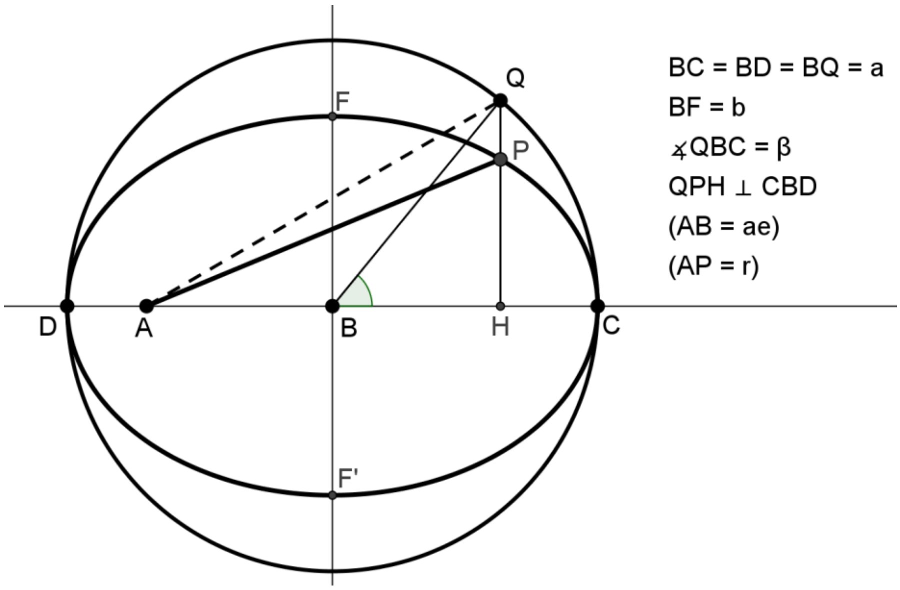

Figure: The Area Law

The ellipse is defined mathematically by the ratio-property of the ordinates, which Archimedes had stated in his work *On Conoids and Spheroids*, Prop.4:

$$\frac{PH}{QH} = \frac{BF}{BC} = \frac{b}{a}. \tag{1}$$

In *Astronomia nova* Chapter 59 Kepler proved geometrically that the area of the ellipse sector PAC determined by the position of the point P can be exactly found, because it is proportional to the associated area *QAC* of the circle. And this gave a value for the time that matched the distance at P, closely in accordance with Tycho Brahe's extremely accurate naked-eye observations.

4. Kepler's proof of that discovery
In the proposition from *On Conoids and Spheroids* already mentioned, Archimedes had proved that:

$$\frac{\textbf{Area of ellipse } \boldsymbol{CPD}}{\textbf{Area of circle } \boldsymbol{CQD}} = \frac{\boldsymbol{b}}{\boldsymbol{a}}\,. \tag{2}$$

Kepler adapted this result to good effect in Chapter 59 Protheorema III. He derived (KGW III p.368 lines 3-4):

$$\frac{\textbf{Ellipse Segment } \boldsymbol{PHC}}{\textbf{Circle Segment } \boldsymbol{QHC}} = \frac{\boldsymbol{b}}{\boldsymbol{a}}\,. \tag{3}$$

Then Kepler applied Euclid's *Elements* VI, 1 (since the two triangles have the same base), to state (KGW III p.368 lines 6-7) that:

$$\frac{\Delta \boldsymbol{PAH}}{\Delta \boldsymbol{QAH}} = \frac{\boldsymbol{b}}{\boldsymbol{a}}\,. \tag{4}$$

Thence, as Kepler said 'by composition' of (3) and (4) – that is, by combining the two pairs of pieces using his diagram above – he derived (KGW III p.368 lines 7-8):

$$\textbf{Ellipse sector } \boldsymbol{PAC} = \frac{\boldsymbol{b}}{\boldsymbol{a}}\ \textbf{Circle sector } \boldsymbol{QAC}\,. \tag{5}$$

This completes Kepler's proof that the area of the ellipse sector which defines the position of the point P can be exactly determined by the associated area of the circle.
(And we add that Kepler had shown much earlier, in Chapter 40 (KGW III p.265 lines 8-10), that area *QAC* can be split into pieces in the following way:

$$\textbf{Circle sector } \boldsymbol{QAC} = \textbf{Sector } \boldsymbol{QBC} + \Delta \boldsymbol{QAB}\,, \tag{6}$$

so that the area of sector *QAC* evidently depends on the angle at the centre of the circle, and can thus be easily evaluated.) Hence the area of the ellipse sector completed by the planet can be expressed in terms of known quantities, and is accordingly itself regarded as known.

We shall now present these expressions of the time taken to reach P in symbolic notation for the benefit of modern readers (writing $\boldsymbol{AB} = \boldsymbol{ae}$):

$$\textbf{Time} \propto \textbf{Circle Sector } \boldsymbol{QAC} = \tfrac{1}{2}\boldsymbol{a}^2.\boldsymbol{\beta} + \tfrac{1}{2}\boldsymbol{ae}.\boldsymbol{a}\sin\boldsymbol{\beta} = \tfrac{1}{2}\boldsymbol{a}^2\,(\boldsymbol{\beta} + \boldsymbol{e}\ \sin\boldsymbol{\beta})\,,$$

or,

$$\textbf{Time} \propto \textbf{Ellipse Sector } \boldsymbol{PAC} = \tfrac{1}{2}\boldsymbol{ab}.\boldsymbol{\beta} + \tfrac{1}{2}\boldsymbol{ae}.\boldsymbol{b}\sin\boldsymbol{\beta} = \tfrac{1}{2}\boldsymbol{ab}\,(\boldsymbol{\beta} + \boldsymbol{e}\ \sin\boldsymbol{\beta})\,.$$

Hence, the variation in time is measured in terms of the angle $\boldsymbol{\beta}$ at the centre:

$$\textbf{Time } \boldsymbol{t} \propto \boldsymbol{\beta} + \boldsymbol{e}\ \sin\boldsymbol{\beta}\,.$$

This expression is often referred to nowadays as 'Kepler's Equation' – he discussed it, with numerical examples, at the end of Chapter 60. But, it is more accurately described as 'Kepler's formula for time in orbit'.

5. The final achievement

However, that is only the first strand of the demonstration. Sharp-eyed readers will have spotted that, while the measure of area has been soundly established, there is apparently no matching theory of time, involving some assumption about how equal amounts of time should be represented geometrically in the diagram. In ancient astronomy time was always represented by an angle, so when Kepler came to deal with this second strand in his mature work of 1621: *Epitome* Book V, Part I, Section 4 (KGW VII) he defined equal small (increment-sized) quantities of time by equal small angles taken at the centre B of the circle. These angles determined equal small circular sectors (increments of area) that Kepler then (using only the geometry of Euclid) manipulated mathematically into equivalence with equal small elliptic sectors round the Sun at A[*]. So eventually Kepler achieved an incremental (near-infinitesimal) proof of the Area law (this mature version is illustrated on numerous websites).

Therefore Kepler's formulation constituted a valid proof that time in orbit was represented by area – but only in an elliptic orbit, when the Sun was the focus and the centre of motion. [The Area-Time proposition in all generality was of course demonstrated by Newton, not until 1687.]

For more details of Kepler's method, see:

Kepler's Planetary Laws

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[*] We cannot say when Kepler recognized that A was the focus of the ellipse – not in *Astronomia nova*, but certainly a decade later when he wrote *Epitome* Book V, since it was a property necessary to that particular proof. (See discussion of Law I for more details.)