

Probes of Fundamental Cosmology (3)

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The Concordance Model

Adopting $H_0 = 73 \text{ km s}^{-1} \text{ Mpc}^{-1}$, we find the following self-consistent set of parameters:

Hubble's constant	$H_0 = 73 \text{ km s}^{-1} \text{ Mpc}^{-1}$
Curvature of Space	$\Omega_\Lambda + \Omega_0 = 1$
Baryonic density parameter	$\Omega_B = 0.04$
Cold Dark Matter density parameter	$\Omega_D = 0.24$
Total Matter density parameter	$\Omega_0 = 0.28$
Density Parameter in Vacuum Fields	$\Omega_\Lambda = 0.72$
Optical Depth for Thomson Scattering on Reheating	$\tau = 0.09$

The remarkable feature of these figures is that they are now known to better than 10% accuracy. This is an extraordinary revolution. We live in the era of **precision cosmology**.

The Properties of the Concordance Model

The scale factor-cosmic time relation for $\Omega_0 + \Omega_\Lambda = 1$ model is

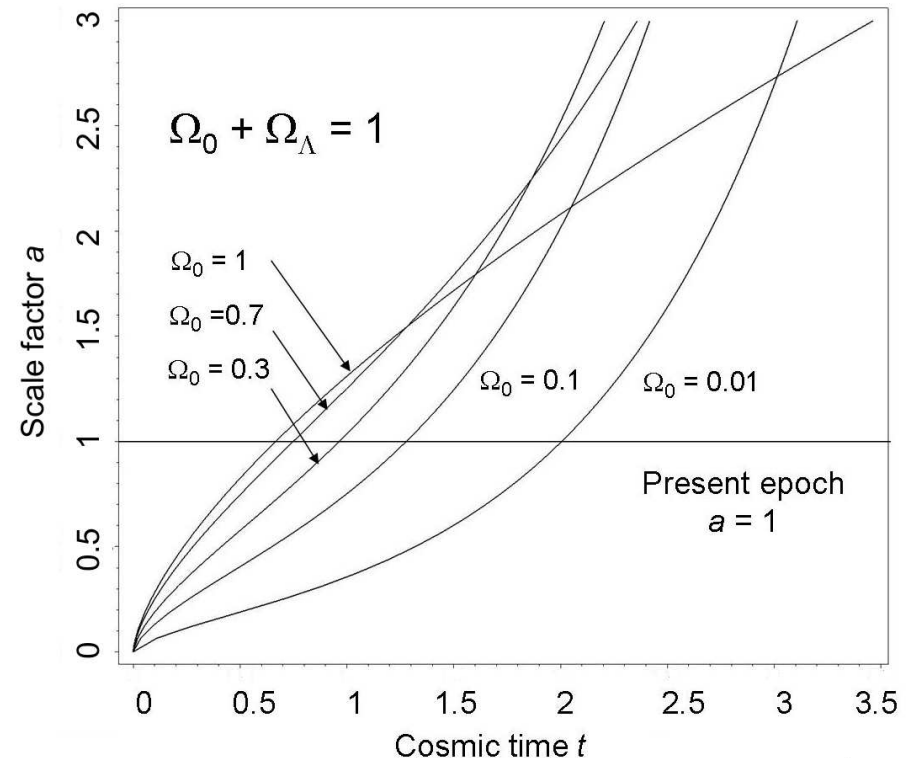
$$t = \int_{t_1}^{t_0} dt = -\frac{1}{H_0} \int_{\infty}^z \frac{dz}{(1+z)[\Omega_0(1+z)^3 + \Omega_\Lambda]^{1/2}},$$

where $a = (1+z)^{-1}$. This results in an accelerating Universe at the present epoch.

If $\Omega_\Lambda = 0.72$ and $\Omega_0 = 0.28$, the age of the world model is

$$T_0 = 0.983 H_0^{-1} = 1.32 \times 10^{10} \text{ years}.$$

Thus, $T_0 \equiv H_0^{-1}$. Is this a coincidence? In the standard Λ CDM model, the answer is **“Yes”**.



Perturbation - Linear and Non-linear

This lecture concerns the physics of the development of perturbations in the standard picture in the linear and non-linear regimes.

- Jeans' Instability in the Expanding Universe
- The Development of the Standard Picture
- Superhorizon scales
- Origin of the Harrison-Zeldovich spectrum (historical)
- The Evolution of the Power-spectrum

The Object of the Exercise

The aim of the cosmologist is to explain how large-scale structures formed in the expanding Universe in the sense that, if $\delta\rho$ is the enhancement in density of some region over the average background density ρ , the *density contrast* $\Delta = \delta\rho/\rho$ reached amplitude 1 from initial conditions which must have been remarkably isotropic and homogeneous. Once the initial perturbations have grown in amplitude to $\Delta = \delta\rho/\rho \approx 1$, their growth becomes non-linear and they rapidly evolve towards bound structures in which star formation and other astrophysical processes lead to the formation of galaxies and clusters of galaxies as we know them.

The density contrasts $\Delta = \delta\rho/\rho$ for galaxies, clusters of galaxies and superclusters at the present day are about $\sim 10^6$, 1000 and a few respectively. Since the average density of matter in the Universe ρ changes as $(1+z)^3$, it follows that typical galaxies must have had $\Delta = \delta\rho/\rho \approx 1$ at a redshift $z \approx 100$. The same argument applied to clusters and superclusters suggests that they could not have separated out from the expanding background at redshifts greater than $z \sim 10$ and 1 respectively.

The Wave Equation for the Growth of Small Density Perturbations

The standard equations of gas dynamics for a fluid in a gravitational field consist of three partial differential equations which describe (i) the conservation of mass, or the equation of continuity, (ii) the equation of motion for an element of the fluid, Euler's equation, and (iii) the equation for the gravitational potential, Poisson's equation.

$$\text{Equation of Continuity} : \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0 ; \quad (1)$$

$$\text{Equation of Motion} : \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} = -\frac{1}{\rho} \nabla p - \nabla \phi ; \quad (2)$$

$$\text{Gravitational Potential} : \nabla^2 \phi = 4\pi G \rho . \quad (3)$$

These equations describe the dynamics of a fluid of density ρ and pressure p in which the velocity distribution is \mathbf{v} . The gravitational potential ϕ at any point is given by Poisson's equation in terms of the density distribution ρ .

The partial derivatives describe the variations of these quantities *at a fixed point in space*. This coordinate system is often referred to as *Eulerian coordinates*.

The Wave Equation for the Growth of Small Density Perturbations

We perturb the system about the uniform expansion $\mathbf{v} = H_0 \mathbf{r}$:

$$\mathbf{v} = \mathbf{v}_0 + \delta \mathbf{v}, \quad \rho = \rho_0 + \delta \rho, \quad p = p_0 + \delta p, \quad \phi = \phi_0 + \delta \phi. \quad (4)$$

After a bit of algebra, we find the following equation for the peculiar velocity induced by the growth of the perturbation:

$$\frac{d}{dt} \left(\frac{\delta \rho}{\rho_0} \right) = \frac{d\Delta}{dt} = -\nabla \cdot \delta \mathbf{v}, \quad (5)$$

where $\Delta = \delta \rho / \rho_0$ is the density contrast. After a bit more algebra, we obtain the wave equation for Δ

$$\frac{d^2 \Delta}{dt^2} + 2 \left(\frac{\dot{a}}{a} \right) \frac{d\Delta}{dt} = \Delta (4\pi G \rho_0 - k^2 c_S^2), \quad (6)$$

where the adiabatic sound speed c_S^2 is given by $\partial p / \partial \rho = c_S^2$ and k is the proper wavevector.

The Jeans' Instability

The differential equation for gravitational instability in a static medium is obtained by setting $\dot{a} = 0$. Then, for waves of the form $\Delta = \Delta_0 \exp i(\mathbf{k} \cdot \mathbf{r} - \omega t)$, the dispersion relation,

$$\omega^2 = c_s^2 k^2 - 4\pi G \rho_0, \quad (7)$$

is obtained.

- If $c_s^2 k^2 > 4\pi G \rho_0$, the right-hand side is positive and the perturbations are oscillatory, that is, they are sound waves in which the pressure gradient is sufficient to provide support for the region. Writing the inequality in terms of wavelength, stable oscillations are found for wavelengths less than the critical *Jeans' wavelength* λ_J

$$\lambda_J = \frac{2\pi}{k_J} = c_s \left(\frac{\pi}{G \rho} \right)^{1/2}. \quad (8)$$

The Jeans' Instability

- If $c_S^2 k^2 < 4\pi G \rho_0$, the right-hand side of the dispersion relation is negative, corresponding to unstable modes. The solutions can be written

$$\Delta = \Delta_0 \exp(\Gamma t + i\mathbf{k} \cdot \mathbf{r}) , \quad (9)$$

where

$$\Gamma = \pm \left[4\pi G \rho_0 \left(1 - \frac{\lambda_J^2}{\lambda^2} \right) \right]^{1/2} . \quad (10)$$

The positive solution corresponds to exponentially growing modes. For wavelengths much greater than the Jeans' wavelength, $\lambda \gg \lambda_J$, the characteristic growth time for the instability is

$$\tau = \Gamma^{-1} = (4\pi G \rho_0)^{-1/2} \sim (G \rho_0)^{-1/2} . \quad (11)$$

This is the famous *Jeans' Instability* and the time scale τ is the typical collapse time for a region of density ρ_0 .

The Jeans' Instability in an Expanding Medium

We return first to the full version of the differential equation for Δ .

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = \Delta(4\pi G\rho - k^2 c_s^2). \quad (12)$$

The second term $2(\dot{a}/a)(d\Delta/dt)$ modifies the classical Jeans' analysis in crucial ways. It is apparent from the right-hand side of (12) that the Jeans' instability criterion applies in this case also but the growth rate is significantly modified. Let us work out the growth rate of the instability in the long wavelength limit $\lambda \gg \lambda_J$, in which case we can neglect the pressure term $c_s^2 k^2$. We therefore have to solve the equation

$$\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{a}}{a}\right)\frac{d\Delta}{dt} = 4\pi G\rho_0\Delta. \quad (13)$$

Let us first consider the special cases $\Omega_0 = 1$ and $\Omega_0 = 0$ for which the scale factor-cosmic time relations are

$$a = \left(\frac{3}{2}H_0 t\right)^{2/3} \quad \text{and} \quad a = H_0 t \quad (14)$$

respectively.

The Jeans' Instability in an Expanding Medium

The Einstein–de Sitter Critical Model $\Omega_0 = 1$. In this case,

$$4\pi G\rho = \frac{2}{3t^2} \quad \text{and} \quad \frac{\dot{a}}{a} = \frac{2}{3t}. \quad (15)$$

and we find the key result

$$\Delta = \frac{\delta\rho}{\rho} \propto a = (1+z)^{-1}. \quad (16)$$

In contrast to the *exponential* growth found in the static case, the growth of the perturbation in the case of the critical Einstein–de Sitter universe is *algebraic*.

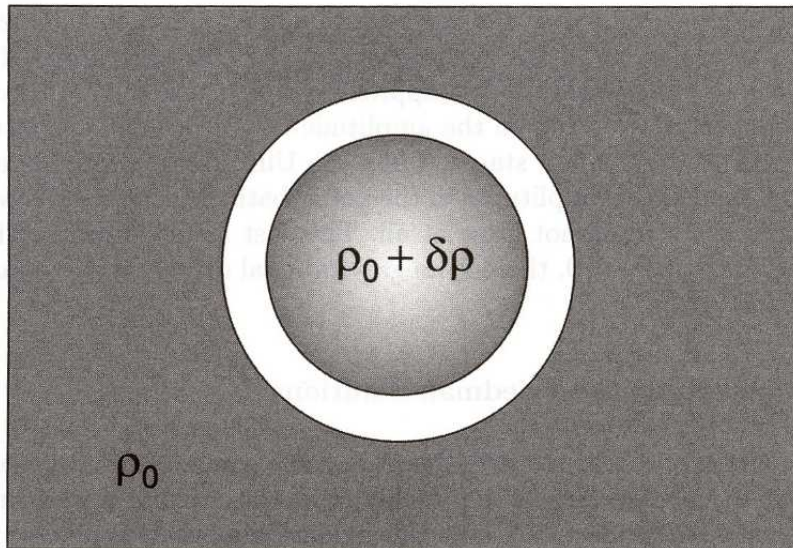
The Empty, Milne Model $\Omega_0 = 0$ In this case,

$$\rho = 0 \quad \text{and} \quad \frac{\dot{a}}{a} = \frac{1}{t}, \quad (17)$$

and $\Delta = \text{constant}$.

In the early stages of the matter-dominated phase, the dynamics of the world models tend to $a \propto t^{2/3}$, and the density contrast grows linearly with a . At redshifts $\Omega_0 z \ll 1$, the amplitudes of the perturbations grow very slowly.

Perturbing the Friedman solutions



Let us derive the same results from the dynamics of the Friedman solutions. The development of a spherical perturbation in the expanding Universe can be modelled by embedding a spherical region of density $\rho + \delta\rho$ in an otherwise uniform Universe of density ρ . The parametric solutions for the dynamics of the world models can be written

$$\begin{aligned} a &= A(1 - \cos \theta) & t &= B(\theta - \sin \theta) ; \\ A &= \frac{\Omega_0}{2(\Omega_0 - 1)} & B &= \frac{\Omega_0}{2H_0(\Omega_0 - 1)^{3/2}} . \end{aligned}$$

Perturbing the Friedman solutions

We now compare the dynamics of the region of slightly greater density with that of the background model. We expand the expressions for a and t to fifth order in θ . The solution is

$$a = \Omega_0^{1/3} \left(\frac{3H_0 t}{2} \right)^{2/3} \left[1 - \frac{1}{20} \left(\frac{6t}{B} \right)^{2/3} \right]. \quad (18)$$

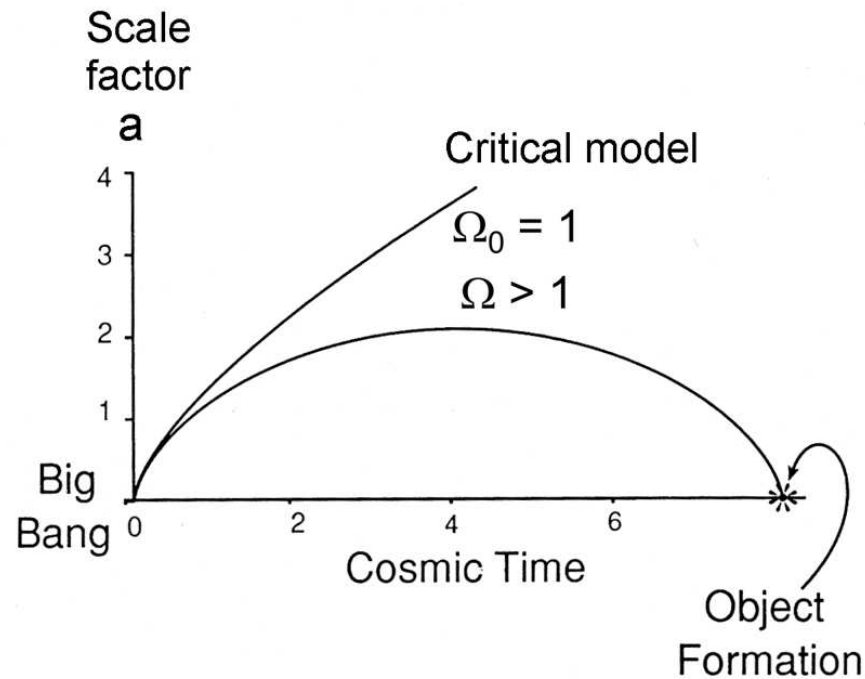
We can now write down an expression for the change of density of the spherical perturbation with cosmic epoch

$$\varrho(a) = \varrho_0 a^{-3} \left[1 + \frac{3(\Omega_0 - 1)}{5 \Omega_0} a \right]. \quad (19)$$

Notice that, if $\Omega_0 = 1$, there is no growth of the perturbation. The density perturbation may be considered to be a mini-Universe of slightly higher density than $\Omega_0 = 1$ embedded in an $\Omega_0 = 1$ model. Therefore, the density contrast changes with scale factor as

$$\Delta = \frac{\delta \varrho}{\varrho} = \frac{\varrho(a) - \varrho_0(a)}{\varrho_0(a)} = \frac{3(\Omega_0 - 1)}{5 \Omega_0} a. \quad (20)$$

Perturbing the Friedman solutions



This result indicates why density perturbations grow only linearly with cosmic epoch. The instability corresponds to the slow divergence between the variation of the scale factors with cosmic epoch of the model with $\Omega_0 = 1$ and one with slightly greater density. This is the essence of the argument developed by Tolman and Lemaître in the 1930s and developed more generally by Lifshitz in 1946 to the effect that, because the instability develops only algebraically, galaxies could not have formed by gravitational collapse.

The General Solutions

A general solution of (12) for the growth of the density contrast with scale-factor for all pressure-free Friedman world models can be rewritten in terms of the density parameter Ω_0 as follows:

$$\frac{d^2 \Delta}{dt^2} + 2 \left(\frac{\dot{a}}{a} \right) \frac{d\Delta}{dt} = \frac{3\Omega_0 H_0^2}{2} a^{-3} \Delta, \quad (21)$$

where, in general,

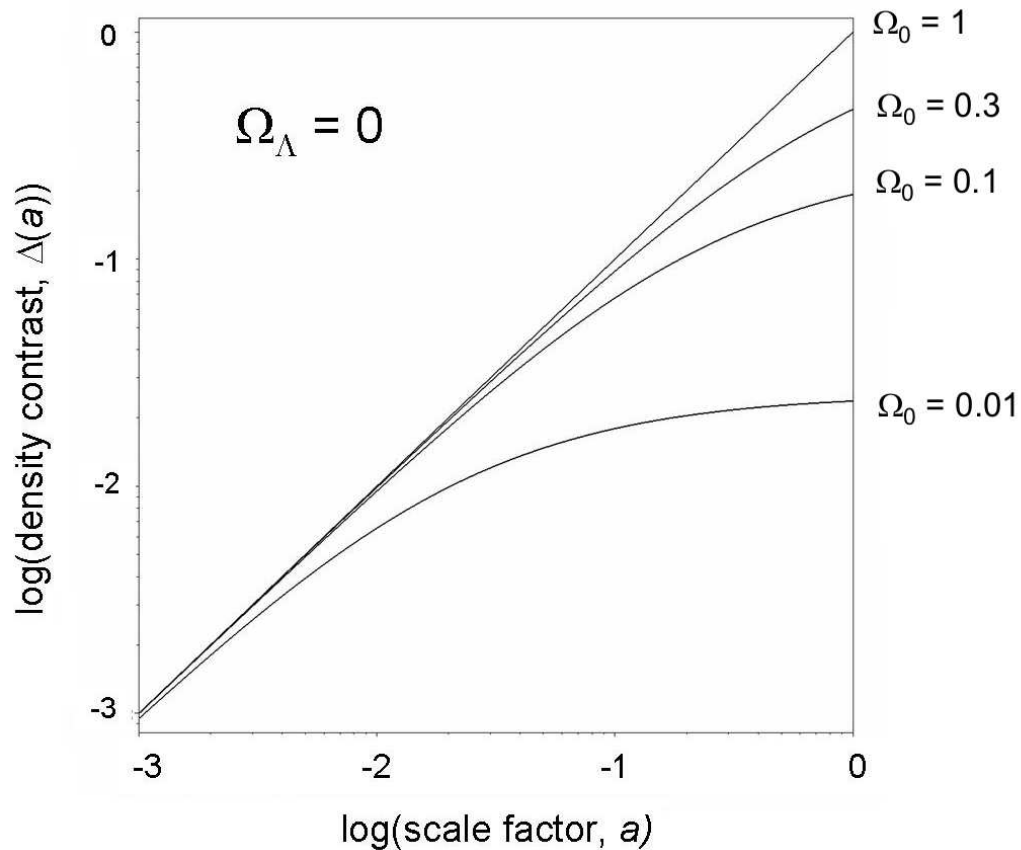
$$\dot{a} = H_0 \left[\Omega_0 \left(\frac{1}{a} - 1 \right) + \Omega_\Lambda (a^2 - 1) + 1 \right]^{1/2}. \quad (22)$$

The solution for the growing mode can be written as follows:

$$\Delta(a) = \frac{5\Omega_0}{2} \left(\frac{1}{a} \frac{da}{dt} \right) \int_0^a \frac{da'}{(da'/dt)^3}, \quad (23)$$

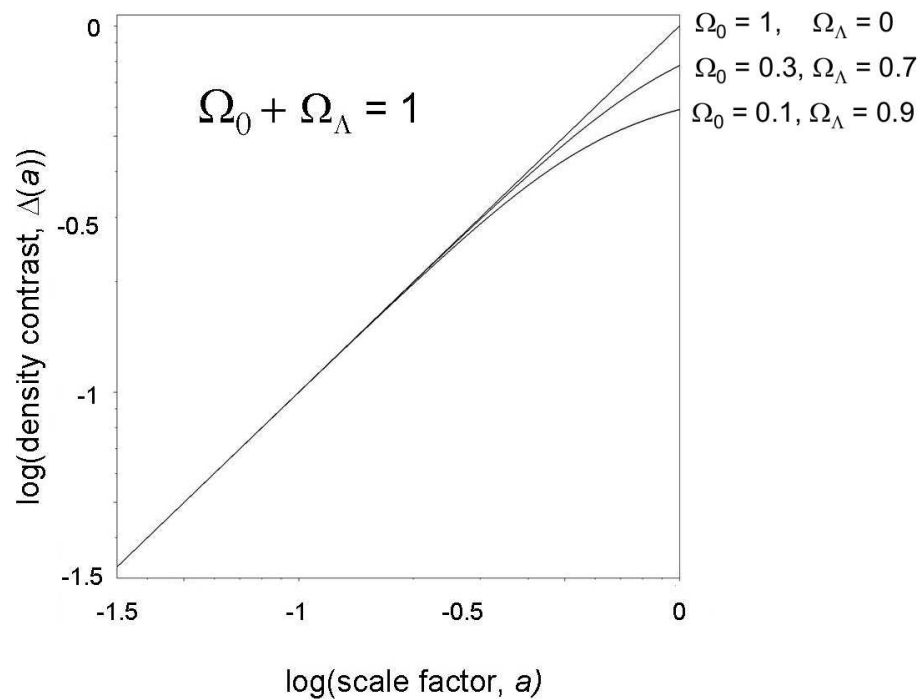
where the constants have been chosen so that the density contrast for the standard critical world model with $\Omega_0 = 1$ and $\Omega_\Lambda = 0$ has unit amplitude at the present epoch, $a = 1$. With this scaling, the density contrasts for all the examples we will consider correspond to $\Delta = 10^{-3}$ at $a = 10^{-3}$. It is simplest to carry out the calculations numerically for a representative sample of world models.

Models with $\Omega_\Lambda = 0$



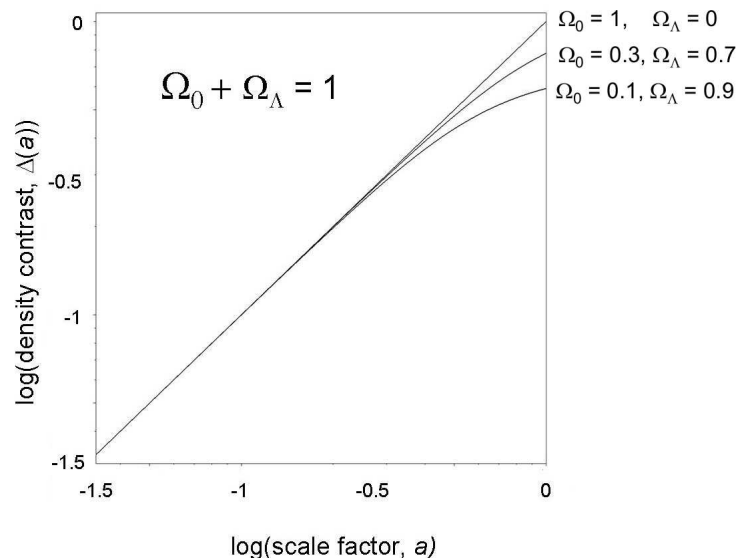
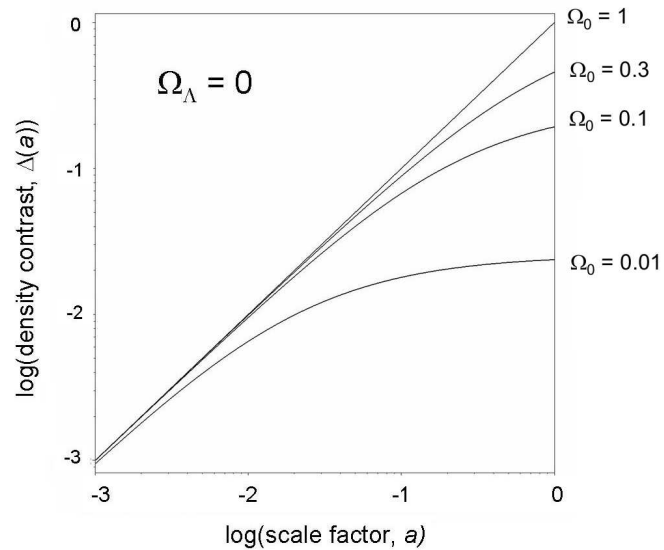
The development of density fluctuations from a scale factor $a = 1/1000$ to $a = 1$ are shown for a range of world models with $\Omega_\Lambda = 0$. These results are consistent with the calculations carried out above, in which it was argued that the amplitudes of the density perturbations vary as $\Delta \propto a$ so long as $\Omega_0 z \gg 1$, but the growth essentially stops at smaller redshifts.

Models with finite Ω_Λ



The models of greatest interest are the flat models for which $(\Omega_0 + \Omega_\Lambda) = 1$, in all cases, the fluctuations having amplitude $\Delta = 10^{-3}$ at $a = 10^{-3}$. The growth of the density contrast is somewhat greater in the cases $\Omega_0 = 0.1$ and 0.3 as compared with the corresponding cases with $\Omega_\Lambda = 0$. The fluctuations continue to grow to greater values of the scale-factor a , corresponding to smaller redshifts, as compared with the models with $\Omega_\Lambda = 0$.

Why are these results so different?



The reason for these differences is that, if $\Omega_\Lambda = 0$, the condition $\Omega_0 z = 1$, corresponds to the change from flat to hyperbolic geometry. This means that neighbouring geodesics are diverging and reduces the strength of the gravitational force.

In the case $\Omega_0 + \Omega_\Lambda = 1$, the geometry is forced to be Euclidean and so the growth continues until the repulsive effect of the Λ term overwhelms the attractive force of gravity. The changeover takes place at much smaller redshifts at $(1+z) \approx \Omega_0^{-1/3}$ if $\Omega_0 \ll 1$.

This is good news if we want to suppress the fluctuations in the Cosmic Microwave Background Radiation.

Peculiar Velocities in the Expanding Universe

The development of velocity perturbations in the expanding Universe can be investigated in the case in which we can neglect pressure gradients so that the velocity perturbations are only driven by the potential gradient $\delta\phi$.

$$\frac{d\mathbf{u}}{dt} + 2 \left(\frac{\dot{R}}{R} \right) \mathbf{u} = -\frac{1}{R^2} \nabla_c \delta\phi . \quad (24)$$

In (24), \mathbf{u} is the perturbed *comoving* velocity and ∇_c is the gradient in comoving coordinates. We split the velocity vectors into components parallel and perpendicular to the gravitational potential gradient, $\mathbf{u} = \mathbf{u}_{\parallel} + \mathbf{u}_{\perp}$, where \mathbf{u}_{\parallel} is parallel to $\nabla_c \delta\phi$. The velocity associated with \mathbf{u}_{\parallel} is often referred to as *potential motion* since it is driven by the potential gradient. On the other hand, the perpendicular velocity component \mathbf{u}_{\perp} is not driven by potential gradients and corresponds to *vortex* or *rotational motions*.

Peculiar Velocities in the Expanding Universe

Rotational Velocities. Consider first the rotational component \mathbf{u}_\perp . The equation for the peculiar velocity reduces to

$$\frac{d\mathbf{u}_\perp}{dt} + 2 \left(\frac{\dot{R}}{R} \right) \mathbf{u}_\perp = 0. \quad (25)$$

The solution of this equation is straightforward $\mathbf{u}_\perp \propto R^{-2}$. Since \mathbf{u}_\perp is a comoving perturbed velocity, the proper velocity is $\delta\mathbf{v}_\perp = R\mathbf{u}_\perp \propto R^{-1}$. Thus, the rotational velocities decay as the Universe expands.

This is no more than the conservation of angular momentum in an expanding medium, $mvr = \text{constant}$. This poses a grave problem for models of galaxy formation involving primordial turbulence. Rotational turbulent velocities decay and there must be sources of turbulent energy, if the rotational velocities are to be maintained.

Peculiar Velocities in the Expanding Universe

Potential Motions. The development of potential motions can be directly derived from the equation

$$\frac{d\Delta}{dt} = -\nabla \cdot \delta v , \quad (26)$$

that is, the divergence of the peculiar velocity is proportional to minus the rate of growth of the density contrast. For the case $\Omega_0 = 1$,

$$|\delta v_{\parallel}| = |Ru| = \frac{H_0 R^{1/2}}{k} \left(\frac{\delta \varrho}{\varrho} \right)_0 = \frac{H_0}{k} \left(\frac{\delta \varrho}{\varrho} \right)_0 (1+z)^{-1/2} , \quad (27)$$

where $(\delta \varrho / \varrho)_0$ is the density contrast at the present epoch. Thus, $\delta v_{\parallel} \propto t^{1/3}$.

The peculiar velocities are driven by both the amplitude of the perturbation and its scale. Equation (27) shows that, if $\delta \varrho / \varrho$ is the same on all scales, the peculiar velocities are driven by the smallest values of k , that is, by the perturbations on the largest physical scales. This is an important result for understanding the origin of the peculiar motion of the Galaxy with respect to the frame of reference in which the Microwave Background Radiation is 100% isotropic and of large-scale streaming velocities.

The Relativistic Case

In the radiation-dominated phase of the Big Bang, the primordial perturbations are in a radiation-dominated plasma, for which the relativistic equation of state $p = \frac{1}{3}\epsilon$ is appropriate.

The equation of energy conservation becomes

$$\frac{\partial \varrho}{\partial t} = -\nabla \cdot \left(\varrho + \frac{p}{c^2} \right) \mathbf{v}; \quad (28)$$

$$\frac{\partial}{\partial t} \left(\varrho + \frac{p}{c^2} \right) = \frac{\dot{p}}{c^2} - \left(\varrho + \frac{p}{c^2} \right) (\nabla \cdot \mathbf{v}). \quad (29)$$

Substituting $p = \frac{1}{3}\varrho c^2$ into (28) and (29), the relativistic continuity equation is obtained:

$$\frac{d\varrho}{dt} = -\frac{4}{3}\varrho(\nabla \cdot \mathbf{v}). \quad (30)$$

Euler's equation for the acceleration of an element of the fluid in the gravitational potential ϕ becomes

$$\left(\varrho + \frac{p}{c^2} \right) \left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p - \left(\varrho + \frac{p}{c^2} \right) \nabla \phi. \quad (31)$$

The Relativistic Case

If we neglect the pressure gradient term, (31) reduces to the familiar equation

$$\frac{dv}{dt} = -\nabla\phi . \quad (32)$$

Finally, the differential equation for the gravitational potential ϕ becomes

$$\nabla^2\phi = 4\pi G \left(\rho + \frac{3p}{c^2} \right) . \quad (33)$$

For a fully relativistic gas, $p = \frac{1}{3}\rho c^2$ and so

$$\nabla^2\phi = 8\pi G\rho . \quad (34)$$

The net result is that the equations for the evolution of the perturbations in a relativistic gas are of similar mathematical form to the non-relativistic case. The same type of analysis which was carried out above leads to the following equation

$$\boxed{\frac{d^2\Delta}{dt^2} + 2\left(\frac{\dot{R}}{R}\right)\frac{d\Delta}{dt} = \Delta\left(\frac{32\pi G\rho}{3} - k^2 c_s^2\right)} . \quad (35)$$

The Relativistic Case

The relativistic expression for the Jeans' length is found by setting the right-hand side equal to zero,

$$\lambda_J = \frac{2\pi}{k_J} = c_s \left(\frac{3\pi}{8G\rho} \right)^{1/2}, \quad (36)$$

where $c_s = c/\sqrt{3}$ is the relativistic sound speed. The result is similar to the standard expression for the Jeans' length.

Neglecting the pressure gradient terms in (35), the following differential equation for the growth of the instability is obtained

$$\frac{d^2\Delta}{dt^2} + 2 \left(\frac{\dot{R}}{R} \right) \frac{d\Delta}{dt} - \frac{32\pi G\rho}{3} \Delta = 0. \quad (37)$$

We again seek solutions of the form $\Delta = at^n$, recalling that in the radiation-dominated phases, the scale factor-cosmic time relation is given by $R \propto t^{1/2}$. We find solutions $n = \pm 1$. Hence, for wavelengths $\lambda \gg \lambda_J$, the growing solution corresponds to

$$\Delta \propto t \propto R^2 \propto (1+z)^{-2}. \quad (38)$$

Thus, once again, the unstable mode grows algebraically with cosmic time.

The Basic Problem of Structure Formation

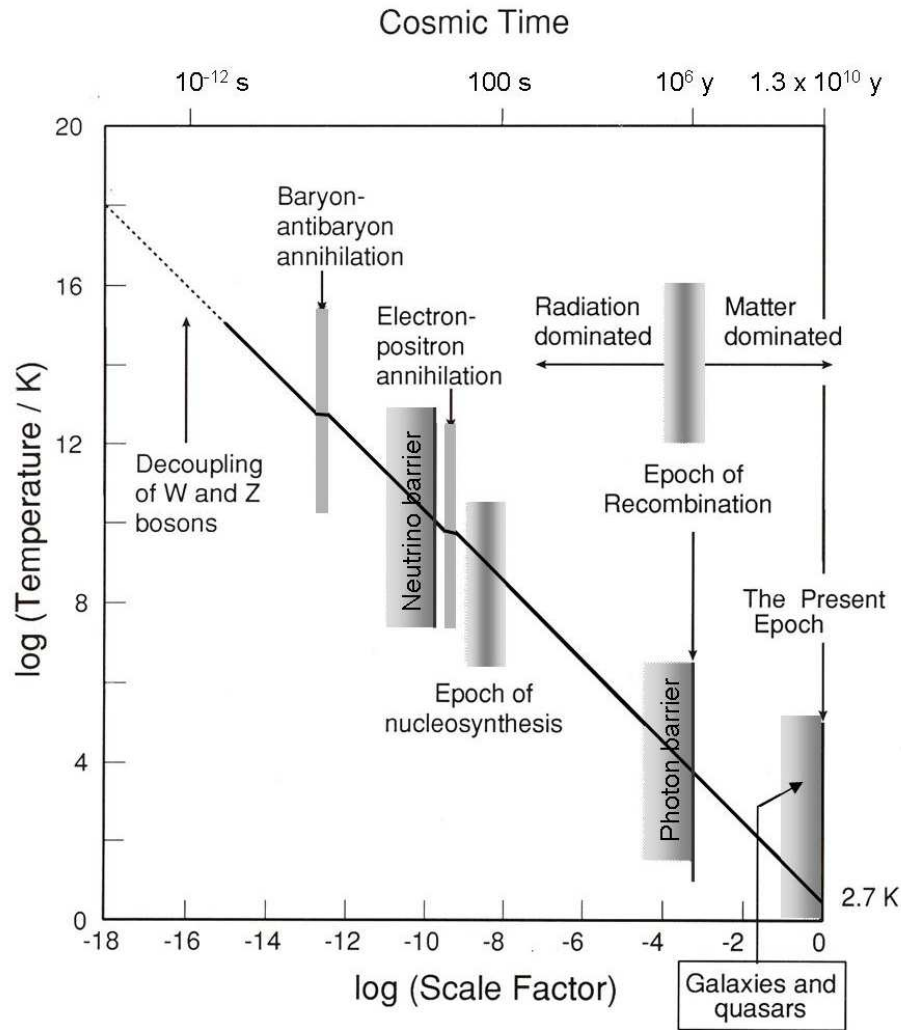
Let us summarise the implications of the key results derived above. Throughout the matter- dominated era, the growth rate of perturbations on physical scales much greater than the Jeans' length is

$$\Delta = \frac{\delta \varrho}{\varrho} \propto a = \frac{1}{1+z}. \quad (39)$$

Since galaxies and astronomers certainly exist at the present day $z = 0$, it follows that $\Delta \geq 1$ at $z = 0$ and so, at the last scattering surface, $z \sim 1,000$, fluctuations must have been present with amplitude at least $\Delta = \delta \varrho / \varrho \geq 10^{-3}$.

- The slow growth of density perturbations is the source of a fundamental problem in understanding the origin of galaxies – large-scale structures did not condense out of the primordial plasma by exponential growth of infinitesimal statistical perturbations.
- Because of the slow development of the density perturbations, we have the opportunity of studying the formation of structure on the last scattering surface at a redshift $z \sim 1,000$.

Summary of the Thermal History of the Universe



This diagram summarises the key epochs in the thermal history of the Universe. The key epochs are

- The epoch of recombination $z = 1000$.
- The epoch of equality of matter and radiation, including neutrinos, $z = 3530$.

The Sound Speed as a Function of Cosmic Epoch

The speed of sound c_s is given by

$$c_s^2 = \left(\frac{\partial p}{\partial \rho} \right)_S, \quad (40)$$

where the subscript S means 'at constant entropy', that is, adiabatic sound waves. The dominant contributors to p and ρ change dramatically as the Universe changes from being radiation- to matter-dominated. The sound speed can then be written

$$c_s^2 = \frac{(\partial p / \partial T)_r}{(\partial \rho / \partial T)_r + (\partial \rho / \partial T)_m}, \quad (41)$$

where the partial derivatives are taken at constant entropy. This reduces to the following expression:

$$c_s^2 = \frac{c^2}{3} \frac{4\rho_r}{4\rho_r + 3\rho_m}. \quad (42)$$

Thus, in the radiation-dominated era, the speed of sound tends to the relativistic sound speed, $c_s = c/\sqrt{3}$.

The Damping of Sound Waves

Although the matter and radiation are closely coupled throughout the pre-recombination era, the coupling is not perfect and radiation can diffuse out of the density perturbations. Since the radiation provides the restoring force for support for the perturbation, the perturbation is damped out if the radiation has time to diffuse out of it. This process is often referred to as *Silk damping*.

At any epoch, the mean free path for scattering of photons by electrons is $\lambda = (N_e \sigma_T)^{-1}$, where $\sigma_T = 6.665 \times 10^{-29} \text{ m}^2$ is the Thomson cross-section. The distance which the photons can diffuse is

$$r_D \approx (Dt)^{1/2} = \left(\frac{1}{3}\lambda ct\right)^{1/2}, \quad (43)$$

where t is cosmic time. The baryonic mass within this radius, $M_D = (4\pi/3)r_D^3 \rho_B$, can now be evaluated for the pre-recombination era.

Particle Horizons and the Hubble Sphere

At early times, all the Friedman models tend toward the dynamics of the critical model and the particle horizon becomes $r_H(t) = 3ct$. This makes physical sense since one might expect that the typical distance which light could travel by the epoch t would be of order ct . The factor 3 takes account of the fact that fundamental observers were closer together at early epochs and so greater distances could be causally connected than ct . A similar calculation can be carried out for the radiation-dominated era and then we find $r_H(t) = 2ct$.

Equally important is the **Hubble sphere**. This is the distance at which $v = c$ according to Hubble's law $v = Hr$ at any epoch, where $H = \dot{a}/a$. This is the distance over which causal phenomena can take place *at a particular epoch*.

Superhorizon Scales

The particle horizon shrinks to vanishingly small values as cosmic, or conformal, time tends to zero. At early enough epochs the horizon becomes smaller than the scales of galaxies, clusters of galaxies and other large scale structures. We cannot avoid tackling the problem of what happens to perturbations on scales greater than the particle horizon, what are called *superhorizon scales*.

On scales less than the horizon scale there was an unperturbed background which acts as a reference frame for the growth of small perturbations. We are also able to synchronise the clocks of all fundamental observers within the particle horizon. If, however, the scale of the perturbation exceeds the horizon scale, what do we mean by ‘the unperturbed background’? Each perturbation then carries its own clock and the whole issue of the synchronisation of clocks and the selection of the appropriate reference frame provide real technical challenges.

This leads to the problem of the *choice of gauge in general relativity*.

Superhorizon Scales

To cut a long story short, one can work in a number of different gauges and, provided one does the sums correctly, one obtains the same result in any gauge. For example, in the *conformal Newtonian gauge* or *longitudinal gauge*, the perturbations are described by the single scalar function ϕ which is just the Newtonian gravitational potential. The metric then has the form

$$ds^2 = a^2(\tau)[(1 + 2\phi)d\tau^2 - (1 - 2\phi)(dx^2 + dy^2 + dz^2)] , \quad (44)$$

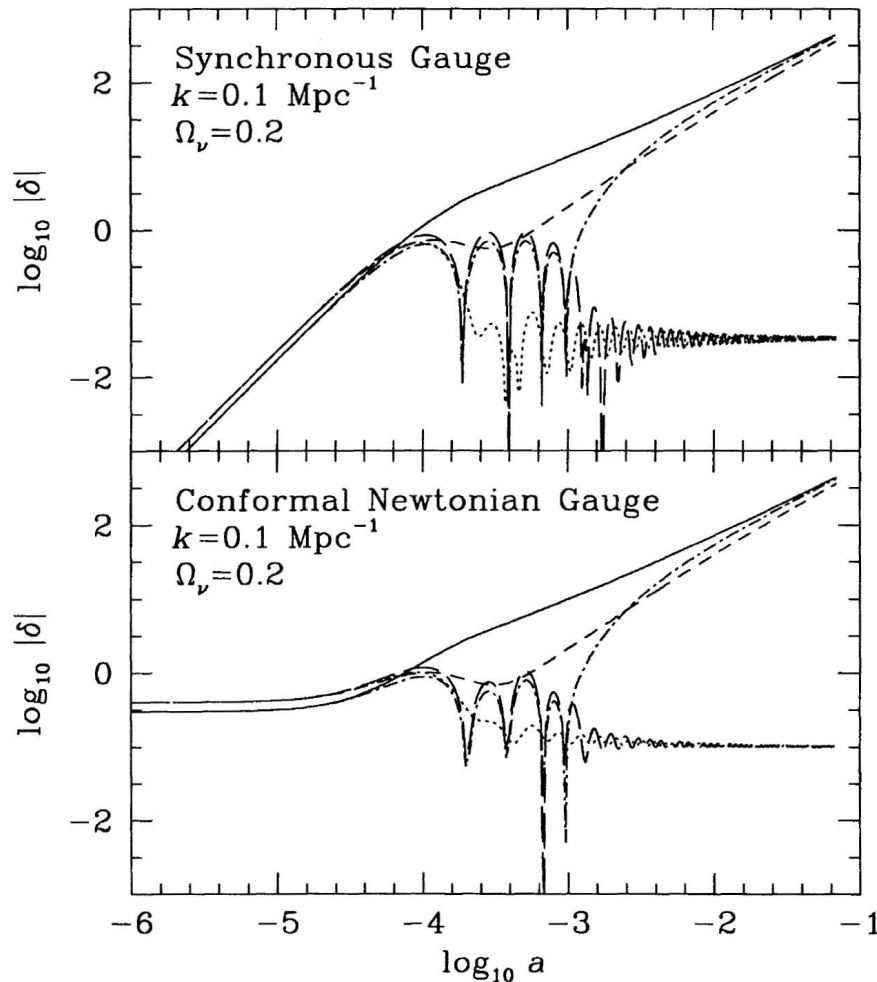
Thus, the Newtonian gravitational potential ϕ provides an accurate description of cosmological perturbations on scales greater than the particle horizon.

An alternative gauge is the *synchronous gauge* in which the term $(1 + 2\phi)$ multiplying $d\tau^2$ is not present, with consequent changes of the other components of the metric. The metric in the synchronous gauge can be written

$$ds^2 = a^2(\tau)\{d\tau^2 - [(1 + 2D)\delta_{ij} + 2E_{ij}] dx^i dx^j\} , \quad (45)$$

This different slicing through space-time illustrates the point that the appearance of the metric depends upon the choice of gauge, although they all contain the same physics in the end.

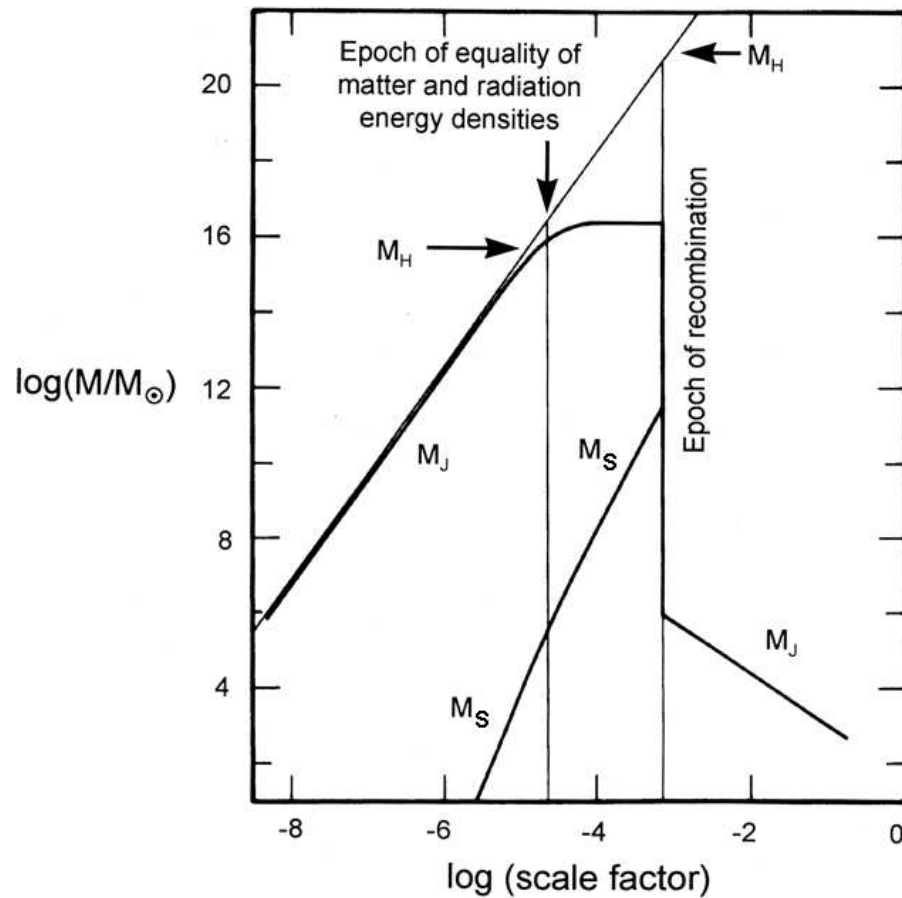
Superhorizon Scales



From Ma and Bertschinger (1995).

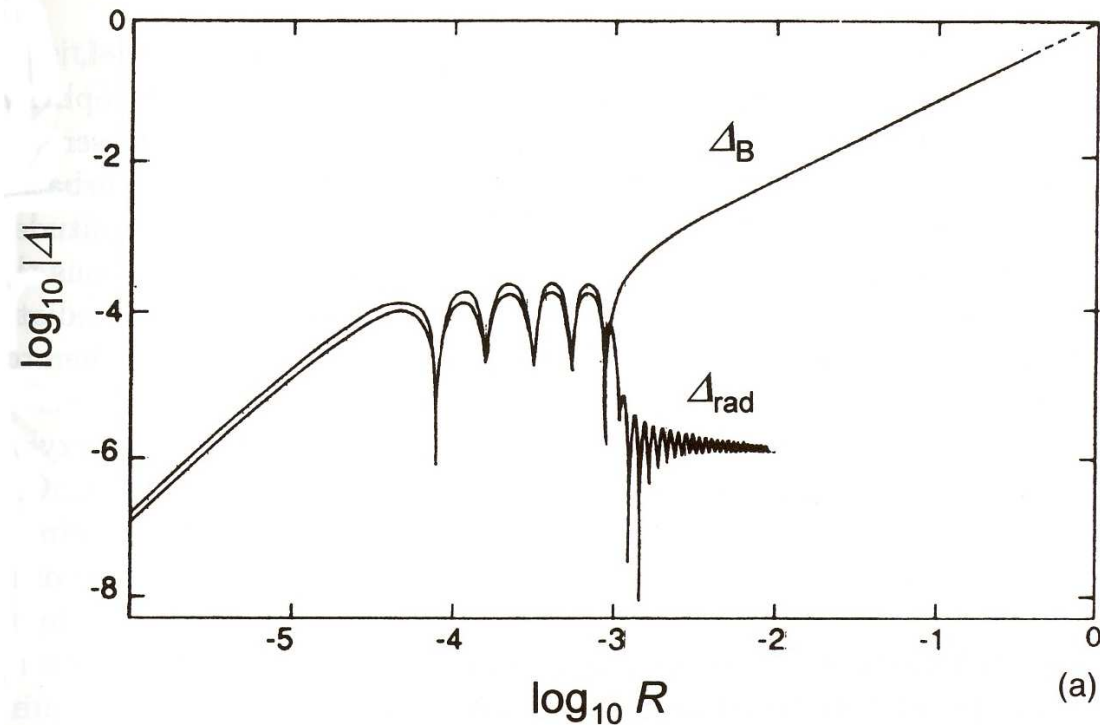
An example of the difference the choice of gauge makes can be seen in the diagram which shows the development of the same set of perturbations in the conformal Newtonian and synchronous gauges. These diagrams reinforce the point that the development of density perturbations can appear very different in the two gauges on superhorizon scales, because of the different slicings through space-time. It can be seen from these diagrams that, once the perturbations come through their particle horizons, the same evolution is found for all five components.

The Simple Baryonic Picture



We can put together all these ideas to develop the simplest picture of galaxy formation. This is the simplest baryonic picture. It includes many of the features which will reappear in the Λ CDM picture. The diagram shows how the horizon mass M_H , the Jeans mass M_J and the Silk Mass M_D change with scale factor a .

The Simple Baryonic Picture



This diagram, from Coles and Lucchin (1995), shows schematically how structure develops in a purely baryonic Universe. The problem is that the temperature fluctuations on the last scattering surface are expected to be at least $\Delta T/T \sim 10^{-3}$, far in excess of the observed limits.

The solution to this problem came with the realisation that the dark matter is the dominant contribution to Ω_0 .

Instabilities in the Presence of Dark Matter

Neglecting the internal pressure of the fluctuations, the expressions for the density contrasts in the baryons and the dark matter, Δ_B and Δ_D respectively, can be written as a pair of coupled equations

$$\ddot{\Delta}_B + 2 \left(\frac{\dot{a}}{a} \right) \dot{\Delta}_B = A \rho_B \Delta_B + A \rho_D \Delta_D, \quad (46)$$

$$\ddot{\Delta}_D + 2 \left(\frac{\dot{a}}{a} \right) \dot{\Delta}_D = A \rho_B \Delta_B + A \rho_D \Delta_D. \quad (47)$$

Let us find the solution for the case in which the dark matter has $\Omega_0 = 1$ and the baryon density is negligible compared with that of the dark matter. Then (47) reduces to the equation for which we have already found the solution $\Delta_D = Ba$ where B is a constant. Therefore, the equation for the evolution of the baryon perturbations becomes

$$\ddot{\Delta}_B + 2 \left(\frac{\dot{a}}{a} \right) \dot{\Delta}_B = 4\pi G \rho_D Ba. \quad (48)$$

Instabilities in the Presence of Dark Matter

Since the background model is the critical model, equation (48) simplifies to

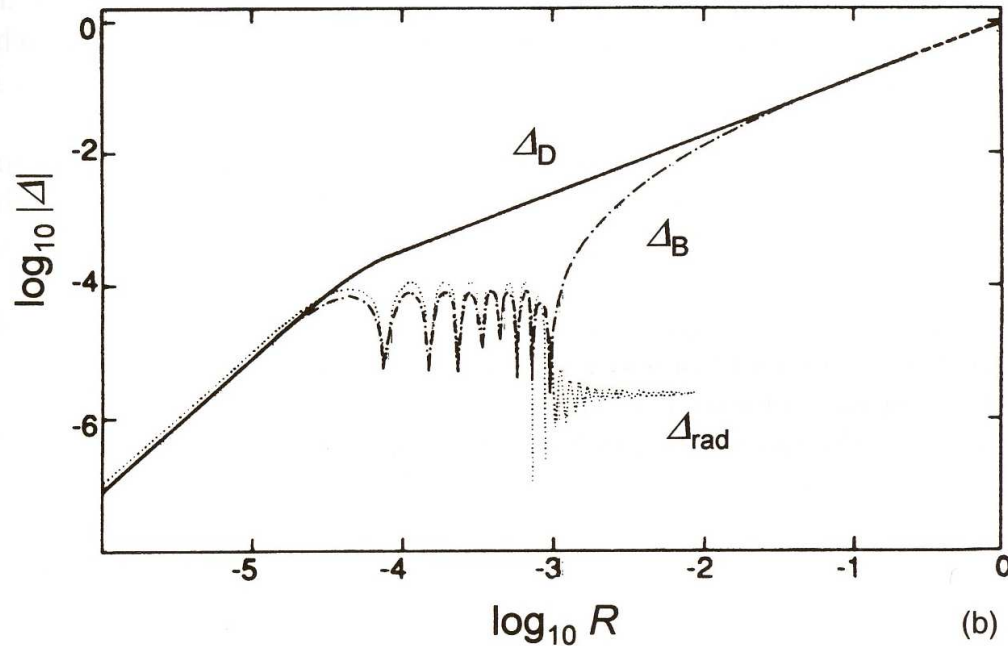
$$a^{3/2} \frac{d}{da} \left(a^{-1/2} \frac{d\Delta}{da} \right) + 2 \frac{d\Delta}{da} = \frac{3}{2} B . \quad (49)$$

The solution, $\Delta = B(a - a_0)$, satisfies (49). This result has the following significance. Suppose that, at some redshift z_0 , the amplitude of the baryon fluctuations is very small, that is, very much less than that of the perturbations in the dark matter. The above result shows how the amplitude of the baryon perturbation develops subsequently under the influence of the dark matter perturbations. In terms of redshift we can write

$$\Delta_B = \Delta_D \left(1 - \frac{z}{z_0} \right) . \quad (50)$$

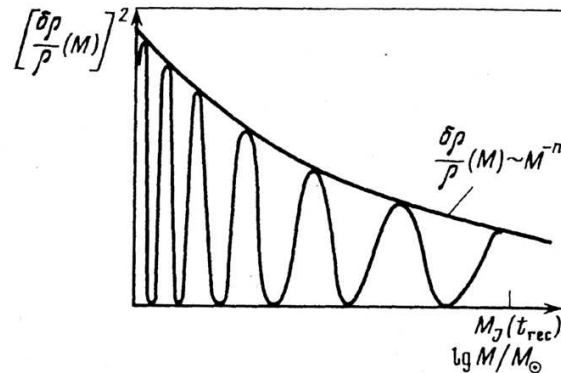
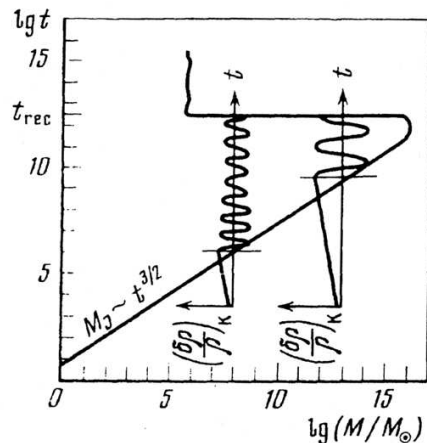
Thus, the amplitude of the perturbations in the baryons grows rapidly to the same amplitude as that of the dark matter perturbations. The baryons fall into the dark matter perturbations and rapidly attain amplitudes the same as those of the dark matter.

The Cold Dark Matter Picture



This diagram shows how structure develops in a cold dark matter dominated Universe. The amplitudes of the baryonic perturbations were very much smaller than those in the cold dark matter at the epoch of recombination.

Note also the origin of the **acoustic peaks** in the predicted mass spectrum (from Sunyaev and Zeldovich 1970).



This is the favoured model for the formation structure.

The Initial Power-Spectrum

The smoothness of the two-point correlation function for galaxies suggest that the spectrum of initial fluctuations must have been very broad with no preferred scales and it is therefore natural to begin with a power spectrum of power-law form

$$P(k) = |\Delta_k|^2 \propto k^n . \quad (51)$$

The correlation function $\xi(r)$ should then have the form

$$\xi(r) \propto \int \frac{\sin kr}{kr} k^{(n+2)} dk . \quad (52)$$

Because the function $\sin kr/kr$ has value unity for $kr \ll 1$ and decreases rapidly to zero when $kr \gg 1$, we can integrate k from 0 to $k_{\max} \approx 1/r$ to estimate the dependence of the amplitude of the correlation function on the scale r .

$$\xi(r) \propto r^{-(n+3)} . \quad (53)$$

Since the mass of the fluctuation is proportional to r^3 , this result can also be written in terms of the mass within the fluctuations on the scale r , $M \sim \rho r^3$.

$$\xi(M) \propto M^{-(n+3)/3} . \quad (54)$$

The Initial Power-Spectrum

Finally, to relate ξ to the root-mean-square density fluctuation on the mass scale M , $\Delta(M)$, we take the square root of ξ , that is,

$$\Delta(M) = \frac{\delta \varrho}{\varrho}(M) = \langle \Delta^2 \rangle^{1/2} \propto M^{-(n+3)/6} . \quad (55)$$

This spectrum has the important property that the density contrast $\Delta(M)$ had the same amplitude on all scales when the perturbations came through their particle horizons, **provided** $n = 1$. Let us illustrate how this comes about.

Before the perturbations came through their particle horizons and before the epoch of equality of matter and radiation energy densities, the density perturbations grew as $\Delta(M) \propto a^2$, although the perturbation to the gravitational potential was frozen-in. Therefore, the development of the spectrum of density perturbations can be written

$$\Delta(M) \propto a^2 M^{-(n+3)/6} . \quad (56)$$

The Initial Power-Spectrum

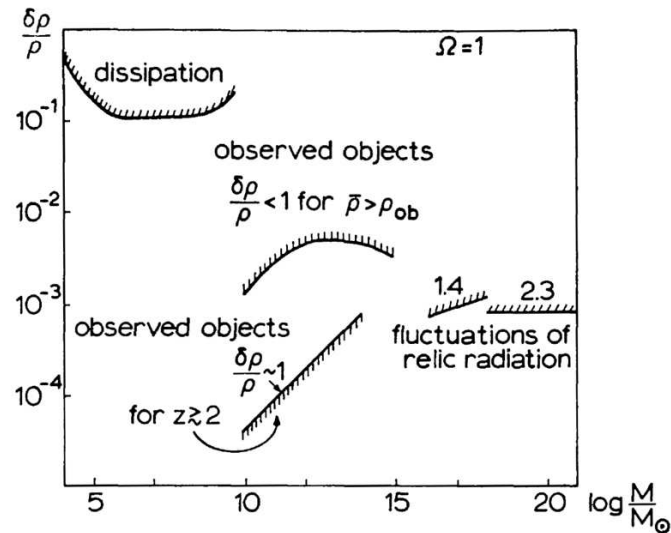
A perturbation of scale r came through the horizon when $r \approx ct$, and so the mass of dark matter within it was $M_D \approx \rho_D (ct)^3$. During the radiation dominated phases, $a \propto t^{1/2}$ and the number density of dark matter particles, which will eventually form bound structures at $z \sim 0$, varied as $N_D \propto a^{-3}$.

Therefore, the horizon dark matter mass increased as $M_H \propto a^3$, or, $a \propto M_H^{1/3}$. The mass spectrum $\Delta(M)_H$ when the fluctuations came through the horizon at different cosmic epochs was

$$\Delta(M)_H \propto M^{2/3} M^{-(n+3)/6} = M^{-(n-1)/6} . \quad (57)$$

Thus, if $n = 1$, the density perturbations $\Delta(M) = \delta\rho/\rho(M)$ all had the same amplitude when they came through their particle horizons during the radiation-dominated era.

The Harrison–Zeldovich Power Spectrum



The constraints on the form of the perturbation spectrum in 1971 derived by Sunyaev and Zeldovich.

They put in what was needed to produce the observed structures today.

Sunyaev and Zeldovich used a variety of constraints to derive the form of the initial power-spectrum of density perturbations as they came through the horizon. They found a scale-invariant spectrum $\delta\rho/\rho = 10^{-4}$ on mass scales from 10^5 to $10^{20} M_\odot$.

Harrison studied the form the primordial spectrum must have in order to prevent the overproduction of excessively large amplitude perturbations on small and large scales. A power spectrum of the form

$$P(k) \propto k \quad (58)$$

does not diverge on large physical scales and so is consistent with the observed large-scale isotropy of the Universe.

Processing of the Initial Power Spectrum

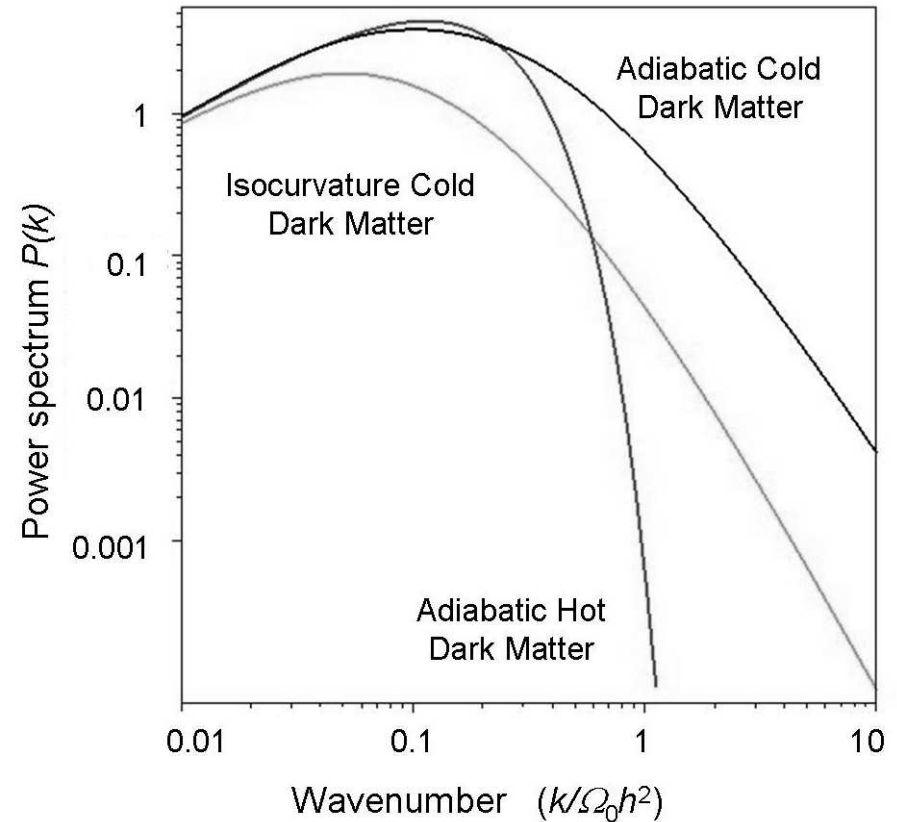
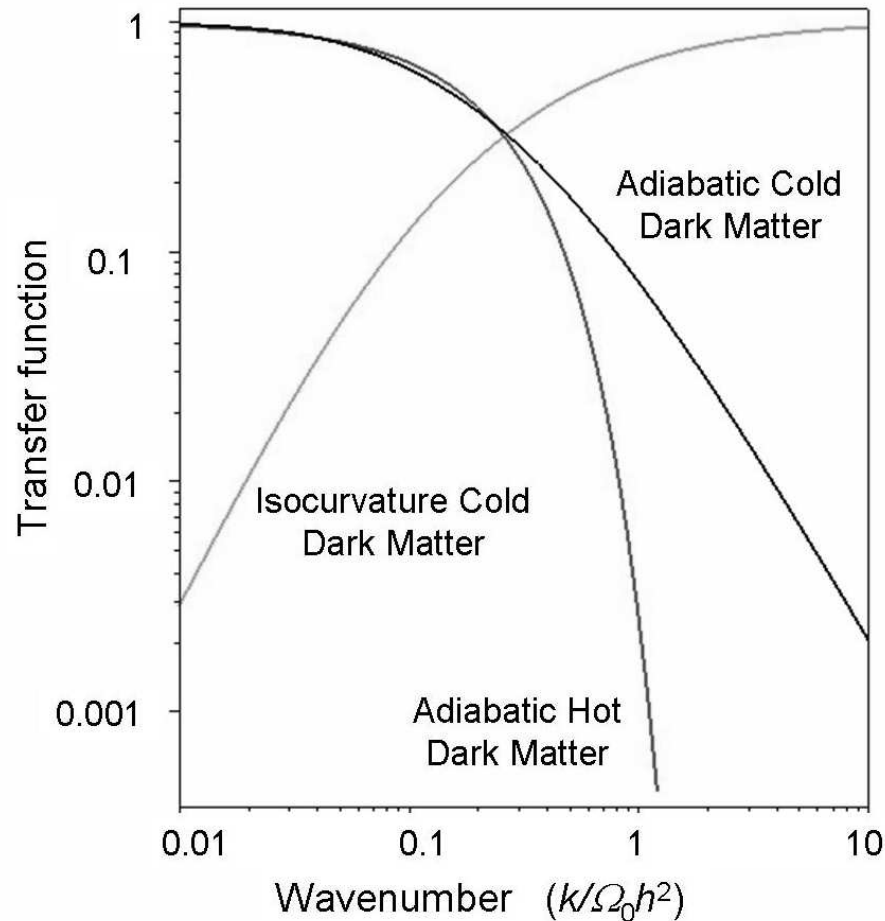
We do not observe the initial power-spectrum except on the largest physical scales. The *transfer function* $T(k)$ which describes how the shape of the initial power-spectrum $\Delta_k(z)$ in the dark matter is modified by different physical processes through the relation

$$\Delta_k(z = 0) = T(k) f(z) \Delta_k(z) . \quad (59)$$

$\Delta_k(z = 0)$ is the power spectrum at the present epoch and $f(z) \propto a \propto t^{2/3}$ is the linear growth factor between the scale factor at redshift z and the present epoch in the matter dominated era.

The form of the transfer function is largely determined by the fact that there is a delay in the growth of the perturbations between the time when they came through the horizon and began to grow again. In the standard cold dark matter picture, this is associated with the fact that before the epoch of equality of matter and radiation, the oscillations in the photon-baryon plasma were dynamically more important than those in the dark matter.

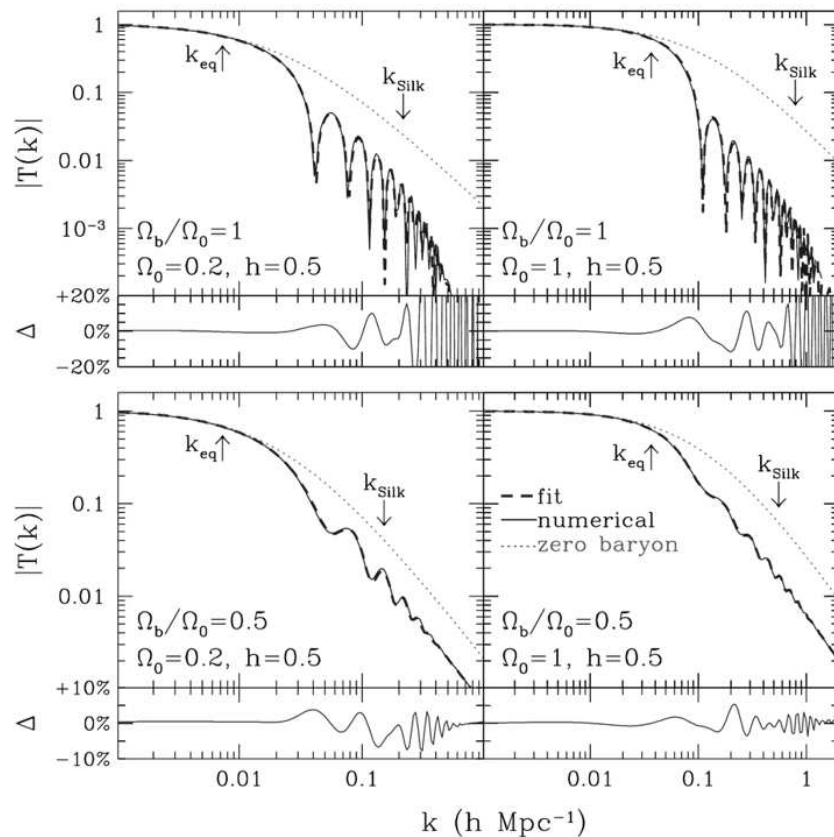
The Processed Harrison–Zeldovich Power Spectrum



Notice that on very large scales (small wavenumbers) the spectrum is unprocessed. On the scale of galaxies and clusters, the spectrum has been strongly modified.

The Processed Harrison–Zeldovich Power Spectrum

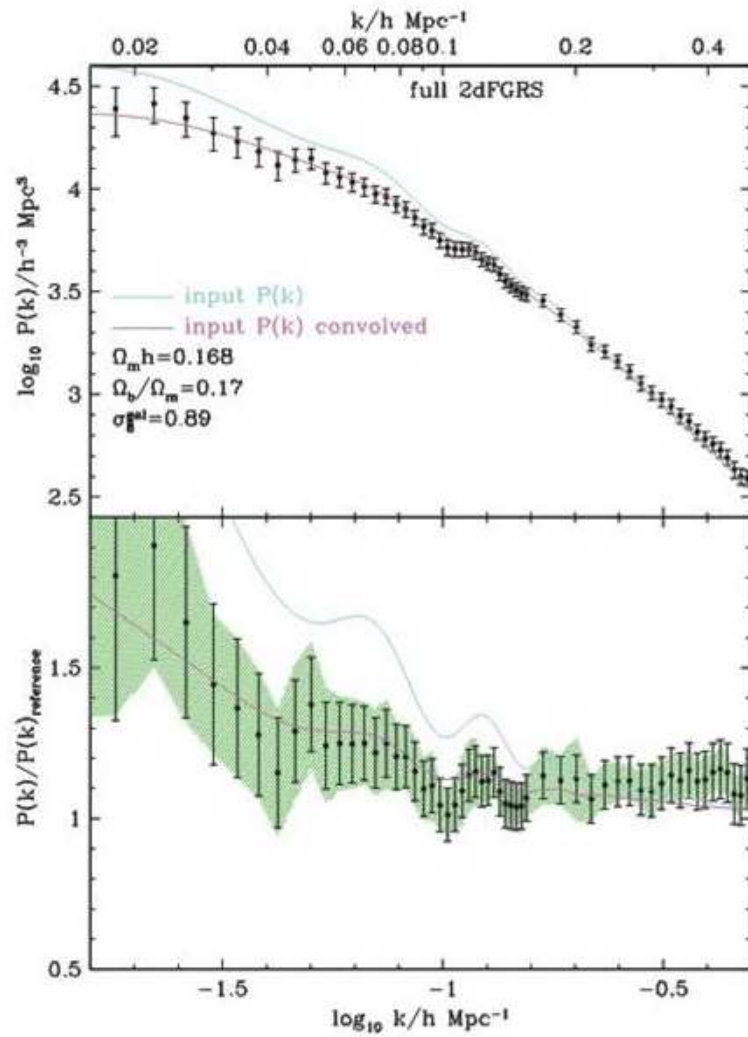
Adding in the Baryons



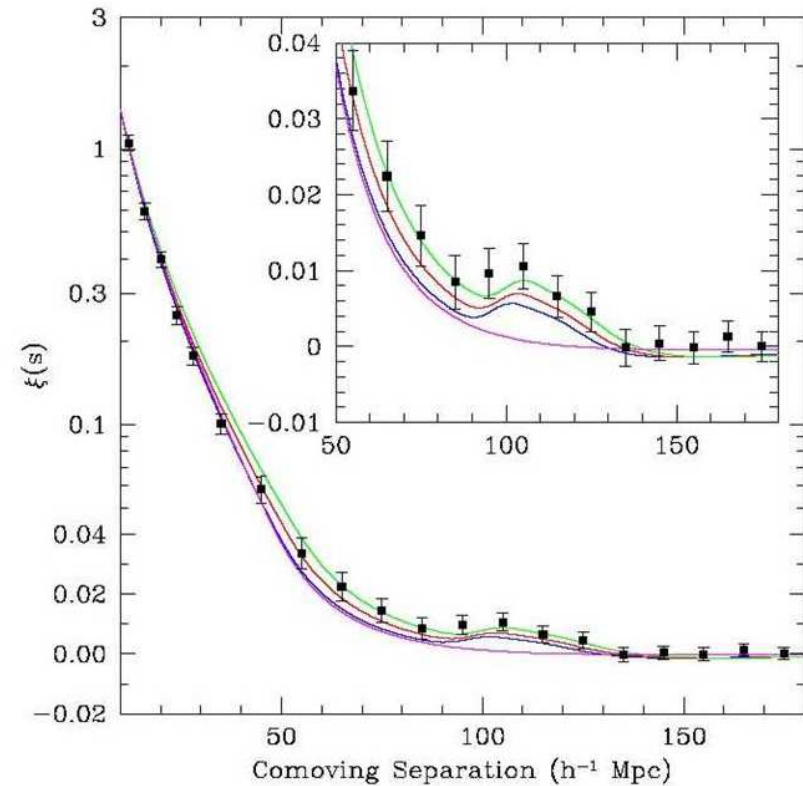
Four examples of the transfer functions for models of structure formation with baryons only (top pair of diagrams) and with mixed cold and baryonic models (bottom pair of diagrams) by Eisenstein and Hu. The numerical results are shown as solid lines and their fitting functions by dashed lines. The lower small boxes in each diagram show the percentage residuals to their fitting functions, which are always less than 10%.

The Acoustic Oscillations in the Galaxy Distribution

AAT 2dF galaxy survey



SDSS galaxy survey



The Basic Input Parameters for the Models

- Selection of a cosmological model with values of Ω_0 , Ω_Λ and H_0 .
- The ordinary baryonic matter has density parameter Ω_B , which is only about 5-10% of the dark matter.
- The power-spectrum of the initial perturbations is assumed to be of Harrison-Zeldovich form $p(k) = Ak^n$ with random phases. The value of n can be varied to find the best fit to the observations.

Run simulation - comoving coordinates