

On an Analytical Framework for Voids: Their abundances, density profiles and local mass functions

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ABSTRACT

We present a general analytical procedure for computing the number density of voids with radius above a given value within the context of gravitational formation of the large scale structure of the universe out of Gaussian initial conditions. To this end we develop an accurate (under generally satisfied conditions) extension of the unconditional mass function to constrained environments, which allows us both to obtain the number density of collapsed objects of certain mass at any distance from the center of the void, and to derive the number density of voids defined by collapsed objects. We have made detailed calculations for the spherically averaged mass density and halo number density profiles for particular voids. We also present a formal expression for the number density of voids defined by galaxies of a given type and luminosity. This expression contains the probability for a collapsed object of certain mass to host a galaxy of that type and luminosity (i.e. the conditional luminosity function) as a function of the environmental density. We propose a procedure to infer this function, which may provide useful clues as to the galaxy formation process, from the observed void densities.

Key words: cosmology:theory — dark matter — galaxies:statistics — large-scale structure of universe — methods:analytical — methods:statistical

1 INTRODUCTION

It is well known that the distribution of galaxies in the Universe is not uniform. The galaxies are distributed in filaments and clusters, leaving large regions devoid of bright galaxies. These regions are known as voids.

The first giant void was the so-called Boötes void found by Kirshner et al. (1981). Subsequently, thanks to large redshift surveys, a large amount of these regions were found and analyzed (e.g. de Lapparent et al. (1986) and Vogeley et al. (1994) in the CfA redshift survey; El-Ad et al. (1997) in the IRAS survey; Müller et al. (2000) in Las Campanas Redshift Survey; Plionis & Basilakos (2002) in the *PSC₂*; Croton et al. (2004) and Hoyle & Vogeley (2004) in the 2dFRGS; Conroy et al. (2005) in the DEEP2 redshift survey)

Many of the theoretical works in the literature about voids are based on cosmological N-Body simulations. The first simulations of the dark matter distribution (Davis et al. 1985) qualitatively showed the existence of large low density regions, but detailed studies of these regions with better res-

olution are just becoming available (Van de Weygaert & Van Kampen 1993; Gottlöber et al. 2003; Colberg et al. 2004 using N-Body simulations and Mathis & White 2002; Benson et al. 2003 using semi-analytical models). Aside from the simulations, there are many analytical works dealing with underdensities in the mass field (see e.g. Sheth & van de Weygaert 2004, and references therein) that give good descriptions of voids. On the other hand, analytical works that define the voids by observable objects (i.e. in point distributions) are not common in the literature (White 1979; Otto et al. 1986; Betancort-Rijo 1990).

An important point about voids is the study of their contents. Despite the word, voids, of course, are not empty. The first detections of galaxies inside voids were spirals near the 'border' of previously defined voids like the Boötes (Dey et al. 1990; Szomoru et al. 1996a and 1996b). However, extrapolating the morphology-density relation (Dressler 1980) one might expect a population of dwarf galaxies well inside the voids. Even though such galaxies have not been observed yet, they will provide, along with the voids statistics, a strong test for the galaxy formation models (Peebles 2001). There is some ongoing progress in the studies of galaxy pop-

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ulations in voids, thanks mainly to the contribution of recent large redshift surveys (see e.g. Rojas et al. 2004; Patiri et al. 2005b).

There are strong discussions about what exactly constitute a void and therefore how to define them. In general, authors define what is a void depending on the studies they are carrying on. In some works, voids are defined as underdensities in a continuous underlying field (Van de Weygaert & Van Kampen 1993; Aikio & Maehoenen 1998; Friedmann & Piran 2001; Sheth & Van de Weygaer 2004; Colberg et al. 2004). In other works, voids are irregular regions delimited by some kind of galaxies, the so-called 'wall' galaxies (e.g. El-Ad & Piran 1997; Hoyle & Vogeley 2002; Benson et al. 2003; Hoyle & Vogeley 2004). Even though, these definitions give a very good idea, for example, about the shape of the galaxy distribution, they do not provide a particularly powerful tool for statistical inference. For this purpose we need a definition which does not smears the information contained in the actual object distribution. Note that if we filter this distribution in certain scale so as to obtain a continuous field and use it to define voids, or classify the objects not by an intrinsic criteria but by a distribution dependent one (e.g. 'wall galaxies'), information is smeared and the ability to discriminate between models by comparing observations with predictions is diminished. This effect is similar to what happens in regard to binned data: the best test using binned data is never better and usually worse than the best test using the raw data.

We define voids as maximal non-overlapping spheres empty of objects with mass above a given one. For example, we could define voids as maximal spheres empty of Milky Way-size halos, so that even though, the voids are empty of these halos, we can have dwarf halos inside the voids (see Fig.(1)). Otto et al. (1986) and Gottlöber et al. (2003) also use this kind of definition.

We will focus our work mainly on rare voids (e.g. the giants ones) because the mean number of these voids is a very sensitive function of the clustering properties of the objects that define those regions. This implies that the available statistics on voids along with more general statistics like the counts in cell moments may be used to obtain accurate information about those clustering properties, providing a powerful tool to discriminate between different large scale structure formation models. To this end the following elements are required: first, a handy framework to compute, for given clustering properties, the mean number of voids defined by certain kind of objects as a function of radius and, second, a precise characterization of the clustering properties which is both easy to use in the framework and physically meaningful.

The aim of this work is to provide these elements and assess their efficiency. We will show how they may be used to infer properties of the processes whereby halos become galaxies of certain kind from the statistics of voids defined by galaxies of that kind. To this end we need to express the void probabilities in terms of the galaxy clustering properties. This can be done in different ways. For example, using all galaxy correlation functions to characterize their clustering properties we could, in principle, obtain the corresponding void statistics (White 1979). However, in practice, this procedure is not feasible. Furthermore, even if it were, the information obtained about the clustering properties in this

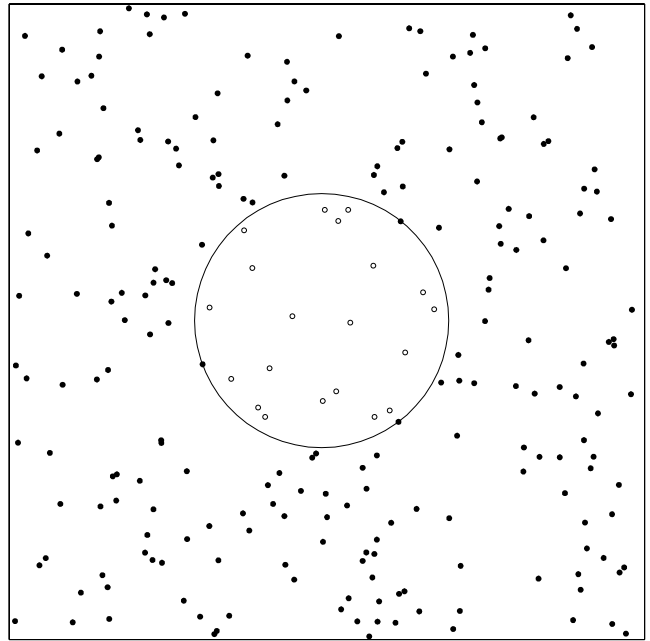


Figure 1. Our definition of voids: maximal non-overlapping spheres empty of objects with mass above M (filled circles). As it is shown, voids can contain objects with masses smaller than M (open circles). Note that we do not show for clarity the smaller object outside the void.

representation do not have a direct connection with galaxy formation processes. In the procedure we present in this work, we first compute the probabilities of voids defined by halos with masses above a given value. This can be done analytically by combining statistics with purely gravitational dynamics. Then, we describe the clustering properties of the galaxies by their relative biasing with respect to halos of the same mass, which may be expressed in terms of the conditional luminosity function, and obtain an expression for the mean number of voids defined by galaxies. For this relative biasing we may either use the existing semi-analytic models (for example, Mo et al. (2004)) using our formalism in a predictive way, or use this formalism in an inductive way to determine that biasing from the observations. This will be presented in a formal way in the discussion, leaving its applications for a future work.

The structure of the work is as follows. In section 2 we use an existing framework (Betancort-Rijo 1990) that allows us to derive the number density¹ of voids of a given radius from the probability that a randomly placed sphere of that radius be empty of the objects defining the void (the VPF). We also show in section 2 how this probability may be obtained by means of an expression containing the biasing of halos with respect to mass. In section 3 we present an extension of the unconditional mass function (UMF) (Sheth and Tormen 1999; Jenkins et al. 2001) to constrained environments, which allow us to obtain the conditional mass function (CMF) which, in turn, is used to quantify the bias-

¹ Technically, the correct term is the *probability density*, which is well defined even when it changes substantially over the local mean distance between voids (or relevant objects). However, to avoid confusion we shall use instead the term: "number density"

ing of halos with respect to mass. In section 4 we show, by combining spherical collapse with the CMF, how to obtain the mean density profile both for the mass and for the halos within and around voids. In section 5 we apply our formalism to different cases to obtain void probabilities and their mass density and halo number density profiles and compare the results with those found in existing numerical simulations. Finally, in section 6 we discuss how to use our formalism to obtain the probabilities of voids defined by galaxies.

2 PROBABILITIES OF VOIDS: THE FRAMEWORK

The general framework for evaluating the mean number densities (i.e. probability densities) of structures like voids (Politzer and Preskill 1986; Otto et al. 1986) has been applied to the standard large scale structure models (i.e. Gaussian initial density fluctuation growing gravitationally). The framework we use here is essentially an updated and extended version of the one developed by Betancort-Rijo (1990). There, the number density, $\bar{P}_0(r)$ of non-overlapping empty (of the objects defining the voids) spheres with radius r is given by:

$$\bar{P}_0(r) = \frac{3\pi^2}{32} \frac{(\bar{n}'V)^3}{V} P_0(r)(1 + \mathcal{O}((\bar{n}'V)^{-1})) \quad (1)$$

with

$$\bar{n}' = \frac{1}{4\pi r^2} \frac{d \ln P_0(r)}{dr}; \quad V = \frac{4}{3} \pi r^3 \quad (2)$$

and

$$P_0(r) = \int_{-1}^{\infty} e^{-(\bar{n}V)(1+\delta_N(\delta))} P(\delta/r) d\delta \quad (3)$$

where $P_0(r)$ is the probability that a randomly placed sphere of radius r be empty (this is the so-called Void Probability Function, VPF), \bar{n} is the mean number density of the objects defining the void, $\delta_N(\delta)$ is the fractional fluctuation of the mean number density of these objects within a sphere of radius r as a function of the actual fractional mass density fluctuation within that sphere, δ , and $P(\delta/r)$ is the probability distribution for the values of δ within a randomly chosen sphere of radius r . The bias of halos with respect to mass is contained within $\delta_N(\delta)$; in the non-biased case (which corresponds to the low halo mass limit) we have simply δ_N equal to δ .

Equation (1) (without the last parenthesis) is valid for rare events, that is, when the mean distance between voids is much larger than the radius, which imply:

$$k \equiv \bar{n}'V \gg 1 \quad (4)$$

In fact, when k is larger than about 3.5, equation (1) with only the zeroth order term in the last parenthesis is sufficiently accurate. Extending eq.(1) to smaller values of k (i.e. obtaining the terms of order k^{-1}) is straightforward (but complex), however, eq.(1) (without the last parenthesis) will be enough to our purposes because the most relevant constraints for galaxy formation models comes from not too common voids (i.e. k sufficiently large).

The exponential in eq.(3) represents the probability that a randomly placed sphere of radius r be empty of the

relevant objects, when the mean fractional density fluctuation within the sphere take the value δ . Multiplying this quantity by the probability for a randomly placed sphere of radius r to have an inner mean fluctuation between δ and $\delta + d\delta$, $P(\delta/r) d\delta$, and integrating over all possible values of δ we obtain $P_0(r)$.

It must be noted that the exponential in Eq.(3) gives correctly the probability that the sphere be empty only when the clustering of the relevant objects conforms to a non-uniform random Poissonian process (Peebles 1980). This is the case to a very high accuracy both when the objects are mass particles or halos of a given mass, on scales (as those of voids) which are much larger than the size of the halos. For galaxies, the model might not be so good. In section 6 we show how to modify expression (3) to be valid for objects whose clustering properties conform to any possible interesting model.

For mass particles $\delta_N = \delta$, so that eq.(3) is particularly simple. From this expression it is easy to see that as the density of particles (\bar{n}) increases, the size of the voids goes to zero.

The probability distribution, $P(\delta/r)$, for a given power spectra is given, for any value of δ in Betancort-Rijo & Lopez-Corredoira (2002). However, since the evolution of large voids is well described by the spherical model, we may use in eq.(3) the spherical approximation to $P(\delta/r)$, also given in that reference with an error that for the voids under consideration is not very relevant, although in some cases, when high accuracy is required, the full $P(\delta/r)$ may be needed. So, we shall use for $P(\delta/r)$:

$$P(\delta/r)d\delta \equiv P(\delta_l/r)d\delta_l = \frac{1}{\sqrt{2\pi}} \frac{\exp\left(-\frac{1}{2} \frac{\delta_l^2}{\sigma^2(r(1+\delta)^{1/3})}\right)}{\left(1 - \left(1 - \frac{\alpha}{2}\right)(1 - (1+\delta)^{-1/3})\right)^{-3}} d\left(\frac{\delta_l}{\sigma(r(1+\delta)^{1/3})}\right) \quad (5)$$

$$\alpha(r) = -\frac{1}{3} \frac{d \ln \sigma^2(r)}{d \ln r} \simeq 0.54 + 0.173 \ln \left(\frac{r}{10h^{-1} Mpc} \right) \quad (6)$$

where $\sigma^2(r)$ is the variance of the linear density fluctuation with a top-hat filtering on a scale r as a function of r ; δ_l is the linear value of the density fluctuations which is related to δ by the spherical model. For $\Omega_m = 0.3$ and $\Omega_\lambda = 0.7$ we have:

$$\delta = D(\delta_l) \equiv 0.993[1 - 0.607(\delta_l - 6.5 \times 10^{-3} \times \times (1 - \theta(\delta_l) + \theta(\delta_l - 1.55))\delta_l^2)^{-1.66} - 1] \quad (7)$$

where

$$\theta(x) = \begin{cases} 1 & \text{if } x > 0 \\ 0 & \text{if } x \leq 0 \end{cases} \quad (8)$$

Using eq.(5) and eq.(7) in eq.(3) and changing the integration variable to δ_l , we have for $P_0(r)$:

$$P_0(r) = \int_{-\infty}^{\infty} e^{-\bar{n}V(1+\delta_N(\delta_l))} P(\delta_l/r) d\delta_l \quad (9)$$

where $\delta_N(\delta_l)$ is the mean fractional fluctuation within r of the number density of the objects defining the void as a

function of the linear fractional density fluctuation within r . Alternatively, integrating over δ we may write:

$$P_0(r) = \int_{-1}^{\infty} e^{-\bar{n}V(1+\delta_N(DL(\delta)))} P(\delta) d\delta \quad (10)$$

where

$$\delta_l \equiv DL(\delta) = \frac{\delta_c}{1.68647} \left[1.68647 - \frac{1.35}{(1+\delta)^{2/3}} - \frac{1.12431}{(1+\delta)^{1/2}} + \frac{0.78785}{(1+\delta)^{0.58661}} \right] \quad (11)$$

This expression for $DL(\delta)$ (Sheth and Tormen 2002) corresponds to the same cosmological parameters as eq.(7) with $\delta_c = 1.676$ and is the inverse function of eq.(7). Note that we write $\delta_N(\delta_l(\delta))$ rather than simply $\delta_N(\delta)$ because it is $\delta_N(\delta_l)$ that we may compute directly (see next section), while $\delta_N(\delta)$ is obtained through the dependence of δ_l on δ . Here we shall use the first expression (Eq.(9)) for $P_0(r)$ while the second must be used when the exact $P(\delta)$, rather than the spherical collapse approximation, is needed. Using eq.(9) (or eq.(10)) in eq.(1) we obtain the mean density of non-overlapping empty sphere of radius r , $P_0(r)$. The number density of voids so that the largest sphere that they can accommodate have radii between r and $r + dr$, $n(r)dr$, is related to $\bar{P}_0(r)$ by:

$$\begin{aligned} \bar{P}_0(r) &= n(\geq r) + n(\geq 2\alpha_1 r) + n(\geq (1 + \frac{2}{\sqrt{3}})\alpha_2 r) + \\ &+ n(\geq (1 + \sqrt{3/2})\alpha_3 r) + n(\geq (1 + 2\sqrt{2/3})\alpha_4 r) + \\ &+ 6n(\geq 2.738\alpha_5 r) + \dots \\ n(> r) &\equiv \int_r^{\infty} n(r) dr \end{aligned} \quad (12)$$

where α_i depends on the mean ellipticity of the voids and may be taken equal to 1 without losing much accuracy. But, in fact, in the interesting cases ($k \gg 1$), $n(2r)/n(r) \ll 1$, so we may write:

$$\bar{P}_0(r) \simeq n(\geq r) \quad ; \quad n(r) \simeq -\frac{d}{dr} \bar{P}_0(r) \quad (13)$$

The mean number, $N(r, \bar{V})$, of voids with radius larger than r within the volume of observation \bar{V} is then:

$$N(r, \bar{V}) = \bar{V} n(> r) \cong \bar{V} \bar{P}_0(r) \quad (14)$$

3 DERIVATION OF $\delta_N(\delta_L)$

3.1 Steps to follow

To derive the mean fractional number density fluctuation within a sphere, δ_N , as a function of the mean linear fractional density fluctuation within it, δ_l , we first obtain the mean fractional fluctuation of the number density of protohalos within the sphere before the sphere expands in comoving coordinates, δ_{ns} , as a function of δ_l . δ_{ns} may be called the statistical fluctuation, since it is due to the clustering of the protohalos in the initial conditions before they move with mass. To obtain $\delta_{ns}(\delta_l)$ we need a framework that enables us to obtain the conditional mass function (CMF) of

collapsed objects $n_c(m, Q, q, \delta_l)$ at a distance q from a point such that the mean density within a sphere of radius Q (radius of the void) centered at this point is δ_l (the linear value; $\delta = D(\delta_l)$ is the actual one). Q and q denote the Lagrangian radius; for Eulerian radius (i.e. present comoving radius) we use respectively R and r . We then have for δ_{ns} :

$$1 + \delta_{ns}(m, Q, \delta_l) = \frac{1}{N_u(m)} \left[\frac{3}{Q^3} \int_0^Q N_c(m, Q, q, \delta_l) q^2 dq \right] \quad (15)$$

$$N_u(m) \equiv \int_m^{\infty} n_u(m) dm \quad (16)$$

$$N_c(m, Q, q, \delta_l) = \int_m^{\infty} n_c(m, Q, q, \delta_l) dm \quad (17)$$

where $n_u(m)$ is the unconditional number density of collapsed object with mass m . The bracketed expression is the mean value within the sphere of radius Q of the number density, N_c , of collapsed objects with masses above m when the mean linear fractional mass density fluctuation within it is δ_l .

As indicated in eq.(15), δ_{ns} depends, in principle, on m , Q and δ_l . However, it may be shown on general grounds (and we have fully checked) that n_c , N_c depend on q and Q almost entirely through q/Q . The reason for this lays in on the goodness of the approximation given in expression (35). As mentioned below this expression, there is a small residual dependence on Q , but it is completely negligible within the relevant range of Q values (less than a factor 2). Then it follows from eq.(15) that δ_{ns} is independent of Q , since, using the change of variable $u = q/Q$, we may write this equation in the form:

$$1 + \delta_{ns}(m, \delta_l) = \frac{1}{N_u(m)} \left[3 \int_0^1 N_c(m, u, \delta_l) u^2 du \right] \quad (18)$$

To obtain the fluctuation in Eulerian coordinates, δ_N , we only need to note that as the void expands the initial fractional halo number density further diminishes by the factor $(1 + \delta)$:

$$1 + \delta_N(m, \delta_l) = (1 + \delta_{ns}(m, \delta_l))(1 + \delta(\delta_l)) \quad (19)$$

This is the expression that we use in eq.(9) to obtain $P_0(r)$. The unconditional mass function of collapsed objects. $n_u(m)$ is given with high accuracy by the unconditional Sheth & Tormen approximation (Sheth & Tormen 1999, 2002; hereafter ST):

$$\begin{aligned} n_{uST}(m, \delta_c, \sigma(m)) &= -\left(\frac{2}{\pi}\right)^{1/2} A \left[1 + \left(\frac{a\delta_c^2}{\sigma^2}\right)^{-p} \right] \\ &\times a^{1/2} \frac{\varrho_b}{m} \frac{\delta_c}{\sigma^2} \frac{d\sigma}{dm} \exp\left(-\frac{a\delta_c^2}{2\sigma^2}\right) \end{aligned} \quad (20)$$

where $A = 0.322$, $p = 0.3$ and $a = 0.707$, ϱ_b stands for the background density, σ is the *rms* linear mass density fluctuation and δ_c is the value of δ_l corresponding to collapse in the spherical model which for the cosmological parameters that we use ($\Omega_m=0.3$, $\Omega_\lambda=0.7$) is 1.676. Our problem now is to obtain a similarly accurate approximation to the conditional mass function, $n_c(m, q, Q, \delta_l)$, so that we can use it in Eq.(18).

3.2 Local constrained mass function

3.2.1 Why do we need a new approach to the CMF ?

Expression (18) gives the Lagrangian constrained accumulated mass function, $N_{cL}(m, \delta_l)$, averaged within a sphere with mean inner linear underdensity δ_l and radius Q (whose dependence on Q has been neglected) given the local Lagrangian accumulated mass function at a distance q from the center of that sphere, $N_c(m, u, \delta_l)$:

$$N_{cL}(m, \delta_l) = N_u(m)(1 + \delta_{ns}(m, \delta_l)) \quad (21)$$

Thus, the accumulated Eulerian mass function averaged within the sphere, N_{cE} , which is the ordinary mass function, is given by:

$$N_c(m, \delta_l) \equiv N_{cE}(m, \delta_l) = N_u(m)(1 + \delta_{ns}(m, \delta_l))(1 + \delta(\delta_l)) \quad (22)$$

These expressions involve an integral within the sphere of the local Lagrangian constrained mass function, $n_c(m, q, Q, \delta_l)$. Thus, this last function is necessary to obtain both $N_c(m, \delta_l)$ and through expression (19), $\delta_N(m, \delta_l)$. There are several approaches (Sheth & Tormen 2002; Gottlöber et al. 2003; Golberg et al. 2004) giving rough approximations and providing reasonable fitting formula for $N_c(m, \delta_l)$, however none of them provides directly (without fitting) an expression sufficiently accurate to allow us to evaluate the void number densities, which depends very sensitively on $\delta_N(m, \delta_l)$. This is due to the fact that the local mass function changes substantially from the center of the sphere ($q = 0$) to its boundary ($q = Q$). In fact, for any δ_l , the mean value of $n_c(m, q, Q, \delta_l)/n_u(m)$ within the sphere is approximately the square root of its value at the center and the value of this quantity at the boundary is almost the cubic root of its value at the center. So, computing the mass function at the center instead of the mean value, or assuming that the interior of the sphere may be replaced by a homogeneous environment characterized by the mean properties of the actual one do not lead to sufficiently good results.

There have been several attempts to derive analytically the CMF. Arguably, the most motivated one is that combining the excursion set formalism (Appel & Jones 1990; Bond et al. 1991) with ellipsoidal dynamics (Sheth & Tormen 2002). However, this procedure is not appropriate to our purposes because, by construction, it gives the CMF at the center of the sphere, which, as we stated before, is quite different from the mean within the sphere, which is the one that we need. In principle, one could repeat the ST derivation at $q = 0$ for any value of q and average over the sphere as indicated in Eq.(9), but this implies a rather complex problem that can not be solved without some approximations. Furthermore, even if the problem could be treated exactly it would provide at most a good fitting formula where some parameters have to be slightly modified (with respect to those given directly by the formalism) to match numerical simulations, as, indeed, have already been done for the unconstrained case (Sheth & Tormen 2002).

In another approach (Gottlöber et al. 2003), the interior of the sphere is treated as a homogeneous environment and the unconditional mass function is rescaled to it. But, leaving aside some queries about the motivation for this procedure, in fact, it disagrees substantially with the simulations for large masses.

Summarizing, neither the available CMF's nor any other we can envisage derived from simple considerations can a priori be expected to give results which are sufficiently accurate to our purposes. Fortunately, although we can not directly obtain analytically a satisfactory CMF, we may analytically extend the UMF through a procedure which is, in practice, exact.

3.2.2 CMF: extending the Unconditional Mass Function

To obtain the CMF, we simply note that, as long as the local evolution at a conditioned point is the same as at an unconditioned one, the conditional local mass function of collapsed objects may be derived from the statistical properties of the local linear field in the same way as the unconditional one.

As to the validity of this assumption, three reasons may be advanced:

- Given the large difference between the scale of the constraint (that of the void) and the scale corresponding to the masses we consider, the conditional shear distribution (of the field filtered on the scale of those masses) can not be very different from the unconditional one

- The profile of the linear density field within the void is not too steep. This means that the mean negative value of the shear in the radial direction imposed by this profile is rather small (it would be strictly zero for a flat profile). So, the departure of the local shear distribution from the unconditional one is smaller than implied in general by the first consideration

- It must be noted that the shear distribution plays a secondary role with respect to the trace (of the local velocity field tensor) in determining the mass distribution of collapsed objects. The difference between ST and the PS formalisms is due to the fact that in the former, the shear distribution is taken into account. The difference between the results of both formalisms is not that large (less than a factor 2). So, the small change in the shear distribution within a void which, according to the two previous considerations, is small, should lead to a negligible error for our extended mass function

That is, if the constrained field behaves “locally” as an isotropic uniform random Gaussian field (with locally defined mean and power spectra), or, alternatively, if the shear distribution of the linear velocity field is at a constrained point equal to that at a randomly chosen one, we may obtain the local mass function using the Unconstrained Mass Function (UMF) for this local field. The relevant statistical property is the probability distribution, $P(\delta_2/\delta_l, q, Q)$, for the linear density fluctuation, δ_2 , on scale Q_2 (that of the halos considered) at a distance q from the center of a sphere of radius Q (the protovoid) with mean inner linear density fluctuation δ_l (see the conceptual diagram in figure (2)).

For a Gaussian field we have:

$$P(\delta_2/\delta, q, Q) = \frac{\exp\left(-\frac{1}{2} \frac{(\delta_2 - \delta_l \frac{\sigma_2}{\sigma_1})^2}{(\sigma_2^2 - \frac{\sigma_2^2}{\sigma_1^2})}\right)}{\sqrt{2\pi} (\sigma_2^2 - \frac{\sigma_2^2}{\sigma_1^2})^{1/2}} \quad (23)$$

where

$$\sigma_1^2 \equiv \langle \delta_1^2 \rangle = \sigma^2(Q); \quad \sigma_2^2 \equiv \langle \delta_2^2 \rangle = \sigma^2(Q_2) \quad (24)$$

$$\sigma^2(x) = \frac{1}{2\pi^2} \int_0^\infty |\delta_k|^2 W^2(xk) k^2 dk \quad (25)$$

$$\sigma_{12} \equiv \sigma_{12}(q, Q, Q_2) = \frac{1}{2\pi^2 q} \int_0^\infty |\delta_k|^2 W(Qk) W(Q_2k) \sin(kq) k dk \quad (26)$$

$$W(x) = \frac{3}{x^3} (\sin x - x \cos x) \quad (27)$$

where $|\delta_k|^2$ is the linear power spectrum of density fluctuations. Comparing with the distribution of δ_2 at a randomly chosen point which is the one implicitly involved in the derivation of the unconditional mass function, we find:

$$P(\delta_2) = \frac{\exp\left(-\frac{1}{2} \frac{\delta_2^2}{\sigma_2^2}\right)}{\sqrt{2\pi}\sigma_2} \quad (28)$$

We see that, at least for the one point statistics, the field δ_2 at a constrained point behaves like an unconstrained field but with a mean value proportional to δ_l and a modified power spectra. It may be shown (Rubiño et al. 2006) that the joint probability distribution for the field δ_2 at several neighboring points follows very closely a Gaussian multivariate with the same mean and power spectra as the one point distribution.

It is then easy to realize that the conditional mass distribution could be obtained through the following substitution in eq.(20) and a renormalization (see Rubiño et al. (2006)):

$$n_c(m, \delta_l, q, Q) \propto n_{uST}(m, \delta'_c, \sigma'(m)) \quad (29)$$

$$\delta'_c = \delta_c - \delta_l \frac{\sigma_{12}(m, q)}{\sigma_1^2} \quad (30)$$

$$\sigma'(m) = \left(\sigma_2^2(m) - \frac{\sigma_{12}^2(m, q)}{\sigma_1^2}\right)^{1/2} \quad (31)$$

σ_2, σ_{12} depends on mass through the mass-scale relationship:

$$Q_2(m) = \left(\frac{m}{3.51 \times 10^{11} h^{-1} M_\odot}\right)^{1/3} h^{-1} \text{Mpc} \quad (32)$$

$$\sigma_2(m) \equiv \sigma(Q_2(m)) \quad (33)$$

$$\sigma_{12}(q, Q, m) \equiv \sigma_{12}(q, Q, Q_2(m)) \quad (34)$$

after this substitution, we obtain the local Lagrangian mass function, $n_c(m, \delta_l, q, Q)$, which as we advanced, in practice, is only a function of q/Q .

Note that this extending procedure is not compromised with any particular fit to the UMF. Actually, we could use for example the fit proposed by Jenkins et al. (2001). For our purposes, however, expressions (12) is to be preferred, because it is very accurate in the mass range we are interested in.

It must be noted that for our expression for the local number density of collapsed objects of mass m to be valid this quantity must change little within a distance of the order of the scale corresponding to m . However, since the scale of variation of the density (for any m) is on the order of the void radius, it is clear that this condition holds provided that $Q_2(m)/Q \ll 1$.

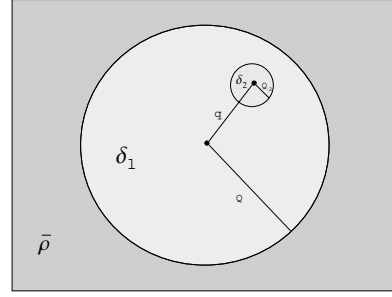


Figure 2. In order to compute the CMF evaluated at certain mass at a distance q from the center of a sphere of radius Q with mean inner linear underdensity δ_1 , we consider the probability distribution at q for the mean linear underdensity (δ_2) within a sphere with radius Q_2 corresponding to the mass under consideration (see text for details).

To carry out the computation we first obtain a fit to $\sigma_{12}(m, q)/\sigma^2(Q)$ and find that it has the form:

$$\frac{\sigma_{12}(q, Q, m)}{\sigma^2(Q)} = c_1 e^{-c_2 \left(\frac{q}{Q}\right)^2} \quad (35)$$

where c_1 and c_2 are certain coefficients almost independent of mass for $Q_2(m) \ll Q$, and only mildly depending on Q . For Q between 3.3 and $6.6 h^{-1} \text{Mpc}$, which include all voids we will consider, c_2 goes from 0.481 to 0.554 . So, we may use the following values for c_1, c_2 over the whole range:

$$c_1 = 1.3212 \quad ; \quad c_2 = 0.525 \quad (36)$$

inserting this in expression 29 and using 18 we find for $\delta_{ns}(\delta_l, m)$:

$$1 + \delta_{ns}(\delta_l, m) = A(m) e^{-b(m)\delta_l^2} \quad (37)$$

$$A(m) \simeq 1 \quad ; \quad b(m) \simeq b'(m)/2 \quad (38)$$

which provides a very good fit for $\delta_l < 0$. $A(m)$ and $b(m)$ are coefficients depending only on mass (for values of Q in the relevant range). Note that we may need different values of c_1, c_2 for voids substantially larger than those considered here. So, the coefficients A, b will, in general, depend on Q as well as on m . The values of $b'(m)$ corresponding to (37) are given in Appendix A.

4 MASS DENSITY AND HALO NUMBER DENSITY PROFILES IN VOIDS

In computing the void number densities we have used, among other things, our CMF and the spherical collapse. Here we shall have the opportunity of checking separately how these assumptions works in explaining the structure of individual voids.

We start with the spherically averaged mass density profile. A given void, characterized by its radius, R , (i.e. that of the largest sphere it can accommodate) and the mean fractional mass density fluctuation within it, δ_0 , has a unique

profile. However, over the ensemble of all voids characterized by these two parameters the density profile varies. What we want to obtain is the mean profile over this ensemble and its dispersion both parameterized by δ_0 and R .

For the rare voids that we shall consider the void density profile is equal to that for a randomly chosen sphere of radius R with inner fractional fluctuation δ_0 . This is so because the fact of whether or not the sphere contains objects (of the type defining the void) can modify the properties of the profile only through the value of δ_0 , which we hold by construction fixed. So, the mean profile within a randomly chosen sphere (with fractional underdensity δ_0) is practically unbiased with respect to that for a void with the same radius and inner underdensity. With this in mind, we may obtain the profile by means of the probability distribution for $\delta(r)$ (the mean fractional enclosed density fluctuation) at a distance r from the center of a randomly chosen sphere with the condition that $\delta(R) = \delta_0$. Transforming to the initial conditions with the relationship of the spherical expansion model $DL(\delta)$ (eq.(10)) between the actual fluctuation, δ , and its linear value, δ_l , our problem is reduced to obtaining the probability distribution, in the initial field, for the value of δ_l at a Lagrangian distance $r(1 + \delta)^{1/3}$ (which transforms into present Eulerian distance r) from the center of a sphere with Lagrangian radius $R(1 + \delta_0)^{1/3}$ and mean inner linear fluctuation $\delta_1 = DL(\delta_0)$. Since the initial conditions are assumed to be Gaussian the distribution of δ_l at a fixed value of q conditioned to $DL(\delta(R)) = DL(\delta_0) \equiv \delta_1$ is immediately given by:

$$P(\delta_l/q, \delta_1) = \frac{1}{\sqrt{2\pi}} \frac{\exp\left(-\frac{1}{2} \frac{(\delta_l - \frac{\sigma_{12}(q)}{\sigma_1^2} \delta_1)^2}{\left(\sigma_2^2 - \frac{\sigma_{12}^2(q)}{\sigma_1^2}\right)}\right)}{\left(\sigma_2^2 - \frac{\sigma_{12}^2(q)}{\sigma_1^2}\right)^{1/2}} \quad (39)$$

$$\sigma_1 \equiv \sigma(Q) \quad ; \quad Q = R(1 + \delta)^{1/3} \quad (40)$$

$$\sigma_2 \equiv \sigma(q) \quad ; \quad q = r(1 + \delta)^{1/3} \quad (41)$$

$$\begin{aligned} \sigma_{12}(q) &= \frac{1}{2\pi^2} \int_0^\infty |\delta_k|^2 W(qk) \\ &\times W(Qk) k^2 dk \end{aligned}$$

Note that this expression gives the conditional probability distribution for δ_l at a fixed q . This is not the conditional probability distribution for the value of δ_l corresponding to the value of δ (through $\delta_l = DL(\delta)$), at some fixed r , which we represent by $\delta(r)$. If it were, we could obtain immediately the conditional distribution for $\delta(r)$ using expression (41) and the relationship between δ and δ_l . The correct derivation of this distribution can be made by a simple (but tedious) argument that we give in Appendix B. We find for the probability distribution for δ (at fixed r) conditioned to $\delta(R) = \delta_0$:

$$P(\delta/r, \delta_1) = \frac{d}{d\Delta} P_c(\Delta) \Big|_{\Delta=\delta} \quad (42)$$

$$P_c(\Delta) = 1 - \frac{1}{2} \operatorname{erfc} \left[\frac{\left| DL(\Delta) - \frac{\sigma_{12}(q)}{\sigma_1^2} \delta_1 \right|}{\sqrt{2} \left(\sigma_2^2 - \frac{\sigma_{12}^2(q)}{\sigma_1^2}\right)^{1/2}} \right] \quad (43)$$

$$\delta_1 = DL(\delta_0) \quad ; \quad q = r(1 + \Delta)^{1/3} \quad (44)$$

with it we have for the mean profile, $\bar{\delta}(r)$:

$$\bar{\delta}(r) = \int_{-1}^\infty \delta P(\delta/r, \delta_1) d\delta = \int_{-1}^\infty (1 - P_c(\Delta)) d\Delta - 1 \quad (45)$$

This integral must extend only to Δ values such that the integrand increases monotonically with Δ . Calling $u(\Delta)$ the argument of erfc in eq.(44) and eq.(45) one may check that the solution to equation:

$$u(\Delta) = u \quad (46)$$

may have more than one branch. A necessary and usually sufficient condition for equation (42) to be valid (see Appendix B) is that a branch, $\Delta^+(u)$, monotonically increasing with u does exist. Other branches correspond to profiles that have experienced a large amount of shell-crossing, so that expression (42) is not valid. In order to account only for the relevant $\Delta^+(u)$ branch the above condition must be imposed upon integral (45).

In an alternative procedure we may lift the mentioned condition on eq.(45) (the first equation) using for $P(\delta/r, \delta_1)$ the absolute value of expression (42) and dividing it into the integral over δ of the absolute value of expression (42), which is larger than 1 when there are additional branches. The difference between this procedure and the former gives a clue as to their accuracy. They are exact only when they agree; otherwise none of them is exact, the former giving a somewhat better result. In a similar way we may obtain $\bar{\delta}^2(r)$, and the profile dispersion $\sigma_\delta(r)$:

$$\sigma_\delta(r) \equiv (\bar{\delta}^2(r) - \bar{\delta}^2(r))^{1/2} \quad (47)$$

As long as the dispersion of the profile is small, which according to the simulations of Gottlöber et al. (2003) holds up to $15h^{-1}\text{Mpc}$ for the $10h^{-1}\text{Mpc}$ void, the mean actual profile should not differ much from the transformed mean linear profile, which we call the most probable profile (in fact, it is very nearly so). Now, from eq.(42) and (44) we see that the most probable profile is essentially given by the center of the Gaussian (in eq.39; i.e. the mean inner profile):

$$\delta_l(q) = \frac{\sigma_{12}(q)}{\sigma^2(Q)} \delta_1 \quad (48)$$

where q and Q are the Lagrangian radius. Transforming δ into δ_l through the spherical model relationship $DL(\delta)$ (equation (11)), and using:

$$\frac{\sigma_{12}(q)}{\sigma^2(Q)} \simeq e^{-c\left(\frac{q}{Q}\right)^2 - 1} \equiv S(q/Q) \quad (49)$$

with

$$c = -\frac{1}{4} \frac{d \ln \sigma^2(Q)}{d \ln Q} \quad (50)$$

we may write:

$$DL(\delta(r)) = \delta_1 S(r(1 + \delta(r))^{1/3}/Q) \quad (51)$$

since

$$q(r) = r(1 + \delta(r))^{1/3} \quad (52)$$

This equation defines implicitly the "most probable" profile $\delta(r, R, \delta_0)$ parameterized by R and δ_0 . This profile is simply the initial mean profile transformed according with

the spherical model. So, although presented in a somewhat different way, the computation is the same as those found in the literature (see e.g. Van de Weygaert & van Kampen 1993).

It follows from eq.(42) that through the following substitution in eq.(51):

$$S(q/Q)\delta_1 \longrightarrow S(q/Q)\delta_1 \pm (\sigma^2(q) - S^2(q/Q)\sigma^2(Q))^{1/2} \quad (53)$$

we can obtain the equations for the upper (+) and lower (-) 68 % confidence level profiles.

The halo number density profiles may now be obtained by combining the mass profiles with equation (37), which gives the fractional fluctuation, $\delta_{ns}(\delta_l)$, of the halo number density, previously to mass motion (i.e. due to the statistical clustering of the protohalos) as a function of the mean value of δ_l within the sphere. The entire fractional fluctuation, δ_N , is given by:

$$1 + \delta_N(r) = (1 + \delta(r))(1 + \delta_{ns}(\delta_l(r))) \quad (54)$$

In the approximation leading to equation (51) (i.e. where we simply transform the mean linear profile) the derivation of $\delta_N(r)$ (most probable value) is particularly simple since, in this case, for a given r there is not only unique δ_N and δ but also unique $\delta_l(r)$. We may then write in eq.(54) the δ value given by eq.(51) and use $DL(\delta(r))$ (expression 11) for $\delta_l(r)$, that is:

$$\delta_N(r) = (1 + \delta(r))A(m)e^{-b(m)(DL(\delta(r)))^2} - 1 \quad (55)$$

$\delta(r)$ being the solution to equation (51) for given values of r , R and δ_0 . This gives the most probable halo number density profile parameterized by R and δ_0 . It must be noted that the full probability distribution for δ_N at a distance r from the center of the void may be obtained through an argument similar to that leading to eq.(42). We find:

$$P(\delta_N/r, \delta(R) = \delta_0) = \frac{1}{2} \frac{d}{d\Delta_N} \text{erfc} \left[\frac{|DL(\Delta_N) - \frac{\sigma_{12}(q)}{\sigma_1^2} \delta_1|}{\sqrt{2} (\sigma_2^2 - \frac{\sigma_{12}^2(q)}{\sigma_1^2})^{1/2}} \right] \Big|_{\Delta_N = \delta_N}$$

$$q = r(1 + D(DL(\Delta_N)))^{1/3} \quad (56)$$

where $\delta_l = DL(\Delta_N)$ is the solution for δ_l as a function of Δ_N of the equation:

$$1 + \Delta_N = (1 + D(\delta_l))(1 + \delta_{ns}(\delta_l)) \quad (57)$$

with $D(\delta_l)$ given by eq.(7) and $\delta_{ns}(\delta_l)$ given by eq.(37). The following relationship:

$$\delta'_N(r) = \frac{1}{3} \frac{1}{r^2} \frac{d}{dr} r^3 \delta_N(r) \quad (58)$$

between the local fractional fluctuation at r , $\delta'_N(r)$, and the average enclosed fluctuation within r , $\delta_N(r)$ (also valid between $\delta'(r)$ and $\delta(r)$) may be used to obtain the profile of local halo number density.

5 RESULTS

5.1 Void counting Statistics

In this subsection, we apply our formalism to compute the mean number densities of voids for several cases, and com-

pare them with those found by Gottlöber et al. (2003) by means of numerical simulations. In order to make a direct comparison we have applied the formalism to the cases treated in the mentioned work.

They carried out high resolution N -Body simulations using the Adaptive Refinement Tree code (ART) of a cube with $80h^{-1}\text{Mpc}$ side. The total number of particles is 1024^3 which leads to a maximum resolution of $4 \times 10^7 h^{-1} M_\odot$ per particle, and a minimum halo mass of $10^9 h^{-1} M_\odot$; the cosmological parameters are $\Omega_m = 0.3$ and $\Omega_\lambda = 0.7$.

They identified voids following a criteria similar to ours; they considered voids as maximal empty spheres in the distribution of dark matter halos (considered as point-like objects). In that simulations, they found that for voids defined by halos with mass larger than $5 \times 10^{11} h^{-1} M_\odot$, the 20 largest voids have radii larger than $7.49 h^{-1} \text{Mpc}$, the 10 largest have radii larger than $9.2 h^{-1} \text{Mpc}$, the 5 largest, $11.0 h^{-1} \text{Mpc}$ and the 3 largest, $11.3 h^{-1} \text{Mpc}$.

On the other hand, when the voids were defined by halos with mass larger than $10^{12} h^{-1} M_\odot$, the 20 largest voids have radii larger than $6.95 h^{-1} \text{Mpc}$, the 10 largest have radii larger than $8.81 h^{-1} \text{Mpc}$, the 5 largest, $11.95 h^{-1} \text{Mpc}$ and the 3 largest, $12.63 h^{-1} \text{Mpc}$. The halo number densities (\bar{n}) were $7.44 \times 10^{-3} (h^{-1} \text{Mpc})^{-3}$ and $4.08 \times 10^{-3} (h^{-1} \text{Mpc})^{-3}$ respectively.

The expected number of voids with radii larger than r within a box of size $L (= 80 h^{-1} \text{Mpc})$, $N(r, L)$, is given by:

$$N(r, L) = \int_r^\infty \bar{V}(r') n(r') dr' \simeq \bar{V}(r) \bar{P}_0(r) \quad (59)$$

$$\bar{V}(r) = (L - 2r)^3 \quad (60)$$

$\bar{V}(r)$ is the available volume for the voids (for their centers) that, since most voids larger than r are only slightly larger than r and expression (15) is a good approximation, the last result follows.

$\bar{P}_0(r)$ is given by expression (1) with $P_0(r)$ given by expression (9). For $(1 + \delta_N)$ we have:

$$(1 + \delta_N) = (1 + \delta)(1 + \delta_{ns}) \quad (61)$$

where δ_{ns} is given by eq. (37) with $A = 1$ and $b(m) = b'(m)/2$. For $b'(m)$ we have used the fit given in Appendix A.

In table 1 we summarize our results and compare them with the results found by Gottlöber et al. (2003).

We have also estimated the size of the largest void expected in the simulation box at the 90 and 68 per cent of confidence level, $\bar{V}(r_0)\bar{P}_0(r_0) = 0.10$ and $\bar{V}(r_0)\bar{P}_0(r_0) = 0.32$ respectively. These results are shown in table 2 along with the largest voids actually found in the simulations.

From these results we may infer that expression (1) gives good results for values of k over 3.5. However, for $N > 7$, regardless of the values of k , our results differ substantially from those found in the simulations. This is due to the fact that the simulation box is small so that it contains only seven or so underdense structures (within which voids are found) rare enough ($|\delta_l|/\sigma \geq 3$) for the spherical collapse to be a good approximation. To obtain good predictions for voids such that $N > 7$ we must use expression (13) rather than eq.(15) and the full expression (Betancort-Rijo & Lopez-Corredoira 2002) for $P(\delta/r)$ must be used in expression (9). However, this will rarely be necessary since

Table 1. Voids: our results vs. simulations. N_p is the Mean number of voids predicted by our formalism within the same volume as the simulations one. N_{sim} is Number of voids found by Gottlöber et al. (2003). P_0 is the VPF and k is the rareness of voids (defined in eq.(4)). The larger the value of k the rarer is the corresponding void. We can see here that as voids become rarer, the analytical predictions are in better agreement with the results from simulations.

| Radius (h^{-1} Mpc) | P_0 | N_p | N_{sim} | k |
|--|-------------|-------|-----------|-------|
| $mass = 5 \times 10^{11} h^{-1} M_\odot$ $\bar{n}=0.00744$ $b=0.0797$ | | | | |
| 11.3 | 0.00101885 | 3.4 | 3 | 4.880 |
| 11.0 | 0.00149288 | 4.8 | 5 | 4.621 |
| 10.8 | 0.00191639 | 6.0 | 7 | 4.471 |
| 9.4 | 0.00984866 | 24.8 | 10 | 3.456 |
| 7.4 | 0.072409 | 119.2 | 20 | 2.227 |
| $mass = 1 \times 10^{12} h^{-1} M_\odot$ $\bar{n}=0.00408$ $b=0.1084$ | | | | |
| 12.6 | 0.000427927 | 1.8 | 3 | 5.434 |
| 11.95 | 0.00107413 | 4.1 | 5 | 4.945 |
| 11.2 | 0.00191895 | 7.4 | 7 | 5.035 |
| 8.8 | 0.0423952 | 70.2 | 10 | 2.756 |
| 7.0 | 0.193217 | 170.3 | 20 | 1.729 |
| $mass = 2 \times 10^{12} h^{-1} M_\odot$ $\bar{n}=0.002162$ $b=0.1389$ | | | | |
| 14.05 | 0.00119284 | 2.2 | 3 | 5.432 |
| 12.8 | 0.00469144 | 6.9 | 5 | 4.432 |
| 11.8 | 0.0125963 | 15.5 | 7 | 3.710 |
| 10.8 | 0.0307436 | 30.7 | 10 | 3.050 |
| 7.5 | 0.296388 | 100.5 | 20 | 1.332 |
| $mass = 5 \times 10^{12} h^{-1} M_\odot$ $\bar{n}=0.000922$ $b=0.3010$ | | | | |
| 16.0 | 0.00541489 | 2.6 | 3 | 4.320 |
| 14.8 | 0.0136554 | 5.7 | 5 | 3.633 |
| 13.5 | 0.0336807 | 11.6 | 7 | 2.957 |
| 9.8 | 0.249094 | 35.2 | 10 | 1.394 |
| 5.8 | 0.849977 | 19.9 | 20 | 0.400 |

the constraints imposed on large-scale structure model by void statistics comes mainly from rare voids.

5.2 Void mass density profiles

In Fig.(3) we show the mean density profile using expression (45) for $R = 10h^{-1}$ Mpc and $\delta_0 = -0.9$, and for $R = 8h^{-1}$ Mpc and $\delta_0 = -0.8667$

In Fig.(4) we present the *most* probable profiles for the cases mentioned above including the 68 % confidence levels for both profiles, this levels define a quite narrow region up to over $13h^{-1}$ Mpc.

Comparing these two figures it is apparent that, although both profiles are very similar within the voids, the mean profile is substantially steeper than the most probable profile at the boundary of the voids. This is due to the asymmetry between the upper and lower one sigma profiles.

Table 2. Largest voids. We can see that the agreement between our predictions and results from numerical simulations is excellent at 90% Confidence Level.

| Mass $h^{-1} M_\odot$ | R_{max} 90% CL | R_{max} 68% CL | R_{max} Simulations |
|--------------------------|---------------------|---------------------|--------------------------|
| 5×10^{11} | 13.8 | 13.0 | 13.0 |
| 1×10^{12} | 14.2 | 13.55 | 13.97 |
| 2×10^{12} | 16.2 | 15.38 | 14.4 |
| 5×10^{12} | 20.04 | 18.6 | 20.03 |

Note that the profiles given here correspond to an average over all empty spheres with quoted δ_0 and R , while those in the simulations correspond to the largest empty sphere with the same δ_0 and R . This implies that the latter profiles should be somewhat steeper than the former ones for r values slightly larger than R (within the sphere, and for r values substantially larger than R they should be equal). It is not difficult to account for this effect, but we shall not consider it here since it is not very relevant.

Comparing with simulations (fig. 3 in (Gottlöber et al. 2003)) we find them to be in very good agreement. In particular, for the $R = 10h^{-1}$ Mpc void, the flatness of the profile within the void with a gentle descent toward the center ($\delta(0) = -0.93$) is found in our results, as well as the steep rise at the boundary. This good agreement strongly suggest that the spherical collapse describes correctly the dynamics of individual rare voids even when their inner underdensity is quite low. That the spherical collapse model may give such good results in several cases, like the present one, where the degree of spherical symmetry is not that high and the tidal field due to the outside matter is not negligible it is an intriguing fact that may be explained by the cancellation of the aspherical effects due to the local matter distribution and the tidal field generated by distant matter (Betancort-Rijo 2004).

It may be checked that for these profiles, both for the most probable one and for the confidence limits, the Lagrangian radius:

$$q(r) = r(1 + \delta(r))^{1/3} \quad (62)$$

is a monotonically increasing function of r . Thus, shell-crossing does not take place and expression (39) applies, so that our procedure is consistent. One could think that this implies that, at least for 68 % of the profiles, shell-crossing does not take place. This is very nearly true, but it must be observed that, in principle, profiles within the limits may have wiggles, so that shell-crossing could be likely to have taken place; although even in this case it will not be very relevant, in the sense that eq.(39) still very nearly applies, for values of r where the confidence region is narrow.

It must also be noted that it is not strictly true that 68 % of the profiles are contained within the 68% confidence region. This is merely the region generated by the confidence intervals for δ at a fixed value of r as r changes. That is, at a given value of r , 68% of the profiles must be within the region although the fraction of profiles that lay within this region for all the r values considered may be somewhat smaller.

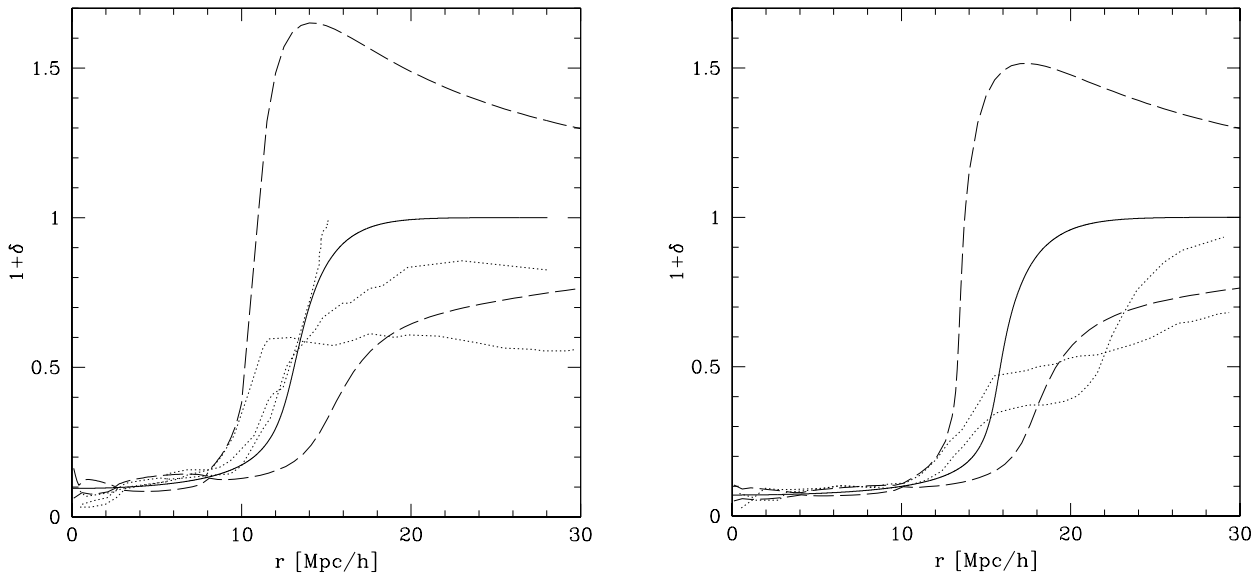


Figure 4. The most probable enclosed density profiles for voids. In the left we show the profile for $R = 8h^{-1}\text{Mpc}$ and $\delta_0 = -0.86667$ and in the right the plot for $R = 10h^{-1}\text{Mpc}$ and $\delta_0 = -0.9$. The solid line, for both, corresponds to the most probable profiles and the dashed line to the 68 % confidence levels. The dotted lines are the profiles found for voids in the numerical simulations.

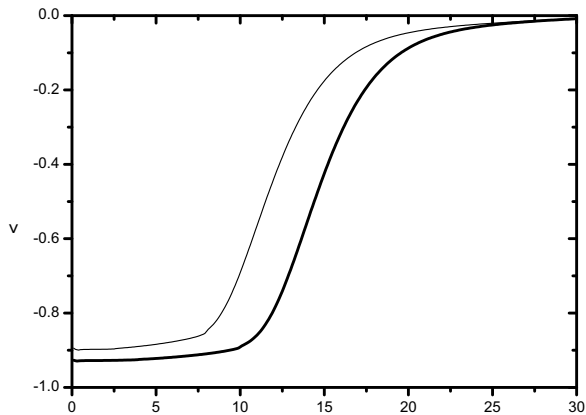


Figure 3. The mean enclosed density profiles for voids. The thin line corresponds $R = 8h^{-1}\text{Mpc}$ and $\delta_0 = -0.8667$ while the thick line to $R = 10h^{-1}\text{Mpc}$ and $\delta_0 = -0.9$

5.3 Halo number density profiles

We have calculated the most probable local halo number density profile, $\delta'_N(r)$, within voids with $R = 8h^{-1}\text{Mpc}$ and $\delta_0 = -0.8667$, for masses above $10^9 h^{-1}M_\odot$ and above $2 \times 10^{10} h^{-1}M_\odot$, which correspond approximately to halos with circular velocity $20\text{km/s} \leq v_{\text{cir}} \leq 55\text{km/s}$ and $55\text{km/s} \leq v_{\text{cir}} \leq 120\text{km/s}$ respectively. To obtain $\delta_N(r)$ we have used Eq.(56) with $\delta(r)$ given by the most probable mass density profile (Eq.(53)) corresponding to the mentioned values of R and δ_0 . $\delta'_N(r)$ has been obtained from $\delta_N(r)$ by means of Eq.(61). In Figure (5) we show the halo number density profiles in and around the void and compared them

with the local mass density most probable profile (obtained from $\delta(r)$ using expression (58)). It is apparent that $|\delta'_N|$ is larger than $|\delta|$ and the more so the larger the mean mass of the halos in the sample. In Figure (6) we show the profiles presented in Fig.(5), but averaging it over five bins of equal volume. Numerical results from Gottlöber et al. (2003) are also shown for comparison. Although this last results corresponds to a superposition of five different values of R and δ_0 the agreement is quite good. Note that using our CMF is essential to explain the results found in the simulations: the ratio between the local halo number density (for masses above $2 \times 10^{10} h^{-1}M_\odot$) at the center and at the boundary is about 2.6, while for the mean mass density this ratio is only 1.54 (for $R = 8h^{-1}\text{Mpc}$ and $\delta_0 = -0.867$). The extra factor 1.69 is due to the different statistical clustering of the protohalos at the center and at the boundary, that is, to the dependence on position of the local number density of protohalos before mass motion (i.e. on Lagrangian coordinates).

In figure (7) we show the halo mass function for two different voids. Note the excellent agreement with simulations (fig.(5) in Gottlöber et al. 2003).

Summarizing, the distribution of matter and halos of given masses within and around voids in simulations may be both reproduced by the combined use of the spherical expansion model and our CMF expression.

6 DISCUSSION

So far in this work we have been dealing only with voids defined by dark matter halos. However, the number density of voids defined by galaxies may be obtained in the same way as those defined by DM halos.

To obtain the number density of the latter we implicitly had to determine the relative biasing of halos above certain mass with respect to the matter. This biasing was responsi-

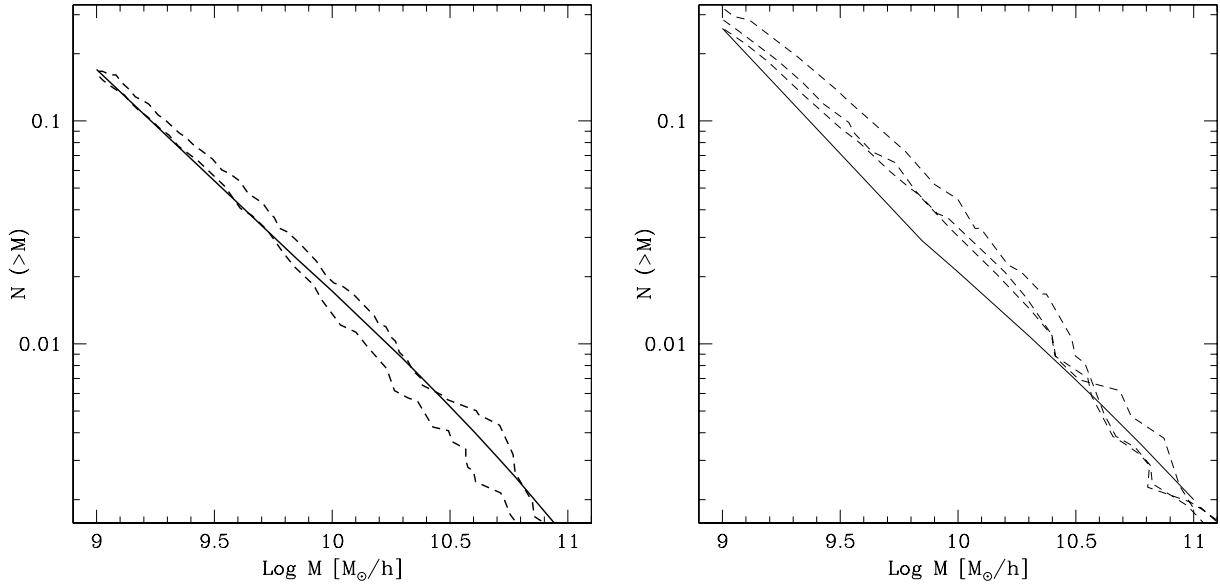


Figure 7. Mass functions of halos averaged within two different voids. Left Panel: The thick full line corresponds to voids with radius $R = 10h^{-1}\text{Mpc}$ and mean density, $\delta_0 = -0.9$ obtained with our formalism. Also in this panel we show the mass function for 2 voids of the same parameters (R, δ_0) obtained from numerical simulations (Gottlöber et al. 2003)(thick dashed line). Right Panel: The thin line denote the mass function for voids with radius $R = 8h^{-1}\text{Mpc}$ and $\delta_0 = -0.86667$. Again, we show results for 3 voids with the same parameters from the numerical simulations. We can see that the agreement of our results with the numerical simulations is very good.

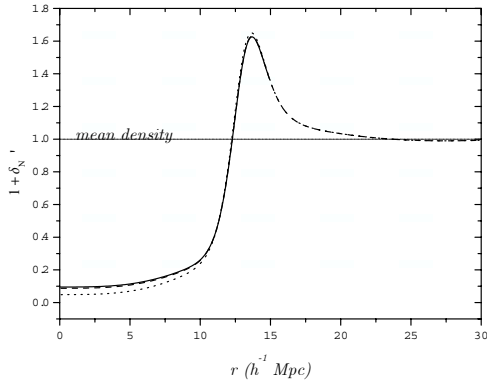


Figure 5. In this plot we show the fractional local halo number density within and around a void with $R = 8h^{-1}\text{Mpc}$ and $\delta_0 = -0.867$. The dashed line corresponds to halos with mass above $10^9 h^{-1}M_\odot$ and the dotted line to those with mass above $2 \times 10^{10} h^{-1}M_\odot$. As a comparison we also show the local mass density profile (full line).

ble for the fact that instead of using in expression (9) $\delta_N = \delta$ which correspond to objects distributed like mass, we had to use:

$$1 + \delta_N = (1 + \delta)(1 + \delta_{ns}) \quad (63)$$

where $1 + \delta_{ns}$, which is due to the initial statistical clustering of the protohalos, accounts for, or rather, is the origin of the biasing of halos with respect to mass.

$1 + \delta_{ns}$ was obtained by studying the dependence of the Lagrangian fractional fluctuation of the number of proto-

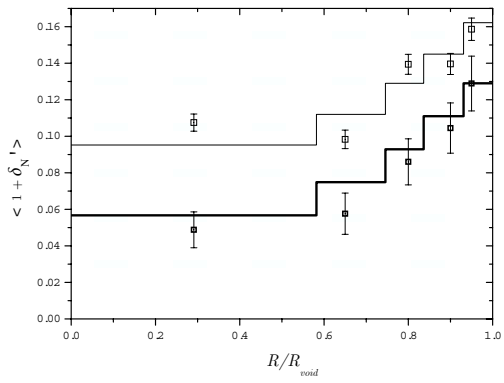


Figure 6. The same as Fig. 4, where $\langle 1 + \delta'_N \rangle$ is the mean halo number density within spherical shells with $1/5$ of the volume of the void. The thin line correspond to halos with mass above $10^9 h^{-1}M_\odot$ and the thick line to halos with mass above $2 \times 10^{10} h^{-1}M_\odot$. R/R_{void} is the distance to the center of the void in units of void radius. The open and half-filled squares correspond to halos with mass above 10^9 and $2 \times 10^{10} h^{-1}M_\odot$ respectively obtained for 5 voids by numerical simulations (Gottlöber et al. 2003). The error bars denote the sampling error.

halos within a sphere on the linear density fluctuation, δ_l , within it.

For voids defined by galaxies of certain type above a given luminosity, L , we must use in expression (63) $1 + \delta_{Ls}$, which describes the initial statistical clustering of the proto-galaxies, instead of $1 + \delta_{ns}$. $1 + \delta_{Ls}$ is obtained by means of the unconditional, $n_u(L)$, and conditional, $n_c(L, q, Q, \delta_l)$, lu-

minosity function in the same way as $1 + \delta_{ns}$ was derived by means of the conditional and unconditional mass functions.

To obtain $n_u(L)$ and $n_c(L)$, we shall first assume that there exist a universal (independent of the environment) Conditional Luminosity Function, $\phi(L/m)$ (Mo et al. 2004, CLF). Then we have:

$$n_u(L) = \int_0^\infty \phi(L/m) n_u(m) dm \quad (64)$$

$$n_c(L, Q, q, \delta_l) = \int_0^\infty \phi(L/m) n_c(m, Q, q, \delta_l) dm \quad (65)$$

Integrating over luminosity from L to infinity we obtain $n_u(> L)$ and $n_c(> L, Q, q, \delta_l)$. Dividing the latter by the former we obtain $1 + \delta_{L_s}(q, Q, \delta_l)$. Averaging over q within the void we finally have:

$$(1 + \delta_{L_s}(Q, \delta_l)) = \frac{3}{Q^3} \int_0^Q \frac{n_c(> L, Q, q, \delta_l)}{n_u(> L)} q^2 dq \quad (66)$$

now, since n_c depends on q and Q almost entirely through q/Q , in practice, δ_{L_s} depends only on δ_l .

So, the number density of voids with radius larger than r defined by galaxies with luminosity larger than L is given by expression (1) with $P_0(r)$ given by eq. (9) and $1 + \delta_N$ given by:

$$1 + \delta_N(\delta_l) = (1 + D(\delta_l))(1 + \delta_{L_s}(\delta_l)) \quad (67)$$

More generally, one might consider the plausible possibility that void galaxies are a systematically different population (Szomoru et al. 1996a; El-Ad & Piran 2000; Peebles 2001; Rojas et al. 2004) or, in mathematical terms, that the Conditional Luminosity Function depends on the environment. Evidence to this dependence on theoretical ground have been recently pointed out by Gao et al.(2005) who reported a dependence of halo clustering on environment through the environmental dependence on halo formation history.

Assuming this dependence enters only through the environmental density we should then use $\phi(L/m, \delta_2)$, where δ_2 stands for the present linear value of the fractional density fluctuation within a sphere centered at the galaxy and with radius $3 - 4h^{-1}\text{Mpc}$ (this radius should be large enough to define a local environment but substantially smaller than the scale length on which these environment change, i.e. large voids).

The only change with respect to the previous case is that now to obtain $n_u(L)$ and $n_c(L)$ expression (25) must be multiplied by the probability distribution for δ_2 at q , $P(\delta_2/q, Q, \delta_1)$, and integrated over δ_2 .

Up to here we have been using a non-uniform Poissonian clustering model: the probability per time unit for a galaxy to form at a given halo may be a function of time and some underlying field, but does not depend on whether or not some galaxies has previously formed in its neighborhood. Dark matter halos formed from Gaussian initial conditions may be shown to obey a Poissonian model. This is a consequence of the validity for them of the peak-background splitting approximation (Bardeen et al. 1986, BBKS). However for galaxies this does not need to be true if the conditional luminosity function depends on the presence of neighboring galaxies: A general model containing these possibilities

is given in Betancort-Rijo (2000). To work within this general model we only need to change $\exp(-\bar{n}V)$ in eq.(1) by $(1 + w\bar{n})^{-V/w}$ where w is an additional parameter to be determined from observations (for Poissonian model $w = 0$).

By means of these expressions we may be able to use void statistics to impose constraints on the possible dependence of $\phi(L/m)$ on environment, and determine the w value.

7 CONCLUSIONS

The extension of the unconditional mass function that we have developed increases very substantially the reach of the original useful tool. Not only is the extension formally exact under the generally satisfied conditions that we have discussed, but it provides the local number density of collapsed objects at any distance from the point at which the constraint is evaluated. This is an essential point for the present work as well as for several other applications, since the fractional number density varies strongly from the center to the boundary of the void (the first value is roughly the cubic root of the second one for any mass). Previous procedures for obtaining the conditional mass function assume that the mass function and the condition are evaluated at the same point. In principle, one could follow those procedures to obtain the mass function at any distance from the center of the void. But this is rather complex and can, at most, provide a good fitting formula. However, as we have shown, once the unconditional mass function has been obtained by deriving a fitting formula through the mentioned procedures (or any other) and calibrating it by means of numerical simulations, its extension to the conditional case can immediately be obtained, without having to repeat the derivation of the fitting formula and its calibration.

This formalism has allowed us to obtain the mean fractional number density of collapsed objects of given mass within a void as a function of the fractional mass density within it. This has been used within the general formalism that we have described here to obtain the number density of voids with radius above a given value. We have compared our results with those found in numerical simulations, checking that for sufficiently rare voids our procedure gives very good results. Furthermore, using $P(\delta/r)$ as given by the spherical expansion/collapse approximation seems to be enough within present uncertainties. Note, however, that we have not checked separately to a sufficient extent the accuracy of the relationship between $P_0(r)$ and $\bar{P}_0(r)$ (Eq.(1)) on the one hand and the accuracy of our computation of $P_0(r)$ on the other. We intend to do this by means of more detailed simulations that will allow us to eliminate some minor uncertainties thereby increasing the accuracy of our procedure.

One relevant issue that we have not addressed here is the redshift distortions. Note that our formalism correspond to real space, while observations are made in redshift space. Recently, a procedure for correcting the observed statistics for redshift distortions has been developed (Patiri et al. 2005). However, in a future work we intend to complement our formalism so that we could make predictions of voids statistics directly in redshift space.

Our formalism provides a simple relation between properties of the galaxy formation process and galaxy distribu-

tion statistics. Using this relation we may infer those properties from the observed distribution. In this manner, in order to assess the consistency of a model of galaxy formation with void statistics it is not necessary to carry out complex numerical simulations with a large dynamical range but only to demand those models to show the properties (i.e. the conditional Luminosity function) inferred through our procedure from void statistics. Furthermore, the effect of the change of any parameter of the model may be estimated immediately.

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APPENDIX A: ALTERNATIVE DERIVATION OF δ_{ns}

In an alternative and more explicit procedure, instead of fitting directly the dependence of δ_{ns} on m and δ_l , we first

obtain the dependence of the fractional halo number density at $q = 0$, which we call δ'_{ns} , on m and δ_l . That is, before dealing with the mean fractional fluctuation within the sphere of radius Q we deal with its value at the center of this sphere. We find that δ'_{ns} may be approximated very accurately (for $\delta_l \leq 0$) by:

$$1 + \delta'_{ns}(\delta_l, m) = A'(m)e^{-b'(m)\delta_l^2} \quad (\text{A1})$$

where for m between $10^9 h^{-1} M_\odot$ and $2 \times 10^{12} h^{-1} M_\odot$:

$$A'(m) \simeq 1 \quad (\text{A2})$$

$$b'(m) = 0.0205 + 0.1155 \left(\frac{m}{3.51 \times 10^{11} h^{-1} M_\odot} \right)^{0.5} \quad (\text{A3})$$

for larger masses we have:

$$A'(m) \simeq 1 \quad (\text{A4})$$

$$b'(m) = 0.1917 + 0.0198 \left(\frac{m}{3.51 \times 10^{11} h^{-1} M_\odot} \right) \quad (\text{A5})$$

To obtain the fractional density fluctuation of the number density of collapsed objects with masses above m , at a distance q from the center of the sphere under consideration, $\delta_{ns}(m, \delta_l, q)$ we simply need to note that:

$$1 + \delta_{ns}(m, \delta_l, q) = \left(1 + \delta'_{ns} \left(m, \frac{\sigma_{12}(m, q)}{\sigma_{12}(m, 0)} \delta_l \right) \right) \times \left(\frac{2 + \frac{\sigma_1^2}{\sigma_2^2} \left(\frac{\sigma_{12}(m, q)}{\sigma_1} \right)^2}{2 + \frac{\sigma_1^2}{\sigma_2^2} \left(\frac{\sigma_{12}(m, 0)}{\sigma_1} \right)^2} \right) \quad (\text{A6})$$

where in the first parenthesis we have noted that in the procedure for extending the UMF expression the dependence on δ_l enters essentially through

$$\frac{\sigma_{12}(m, q)}{\sigma^2(Q)} \delta_l \quad (\text{A7})$$

which as we have seen (eq.35) is independent of m . The second parenthesis accounts for the differences between the expressions that substitute $d \ln \sigma / dm$ in the extensions of UMF expression at $q = 0$ and at a given q . This parenthesis is, for the values we shall use, very close to 1 and may be neglected. Having $\delta_{ns}(m, \delta_l, q)$ at any q we may immediately obtain $\delta_{ns}(m, \delta_l)$ by computing its mean value within the sphere:

$$1 + \delta_{ns}(m, \delta_l) = 3 \int_0^1 A'(m) e^{-b'(m)\delta_l^2 e^{-1.05u^2}} u^2 du \quad (\text{A8})$$

here, we have used eq.(18) and eq.(A1), setting $u \equiv q/Q$. We find again that this expression may be accurately fitted by eq.(A1) with $A(m) \simeq A'(m) \simeq 1$ and $b(m)$ approximately equal to $b'(m)/2$ in all the range of masses considered. δ_{ns} is the mean (within Q) fractional fluctuation of the number density in Lagrangian coordinates (i.e., before the halos move along with mass). This fluctuation is due to the statistical clustering, in the initial field, of the protohalos under consideration (those with mass above m).

APPENDIX B: DERIVATION OF $P(\delta/R, \delta_1)$

To obtain expression (42) for the probability distribution for the mean value of δ within r , we compute first the probability that $\delta \leq \Delta$ (where Δ is some given value), which we represent by $P_c(\Delta)$. If at a certain q value, q_0 , $\delta_l(q_0)$ (the linear value of the fractional density fluctuation within Lagrangian radius q) were equal to $DL(\Delta)$ (the linear value corresponding through spherical collapse to an actual value Δ), then the sphere concentric with the void and with Lagrangian radius q_0 show a mean inner fractional fluctuation equal to Δ corresponding to an expansion by a factor $(1 + \Delta)^{-1/3}$ and, if its present radius is r , we must have $q_0 = r(1 + \Delta)^{1/3}$. If $\delta_l(q_0)$ were less than $DL(\Delta)$ then the actual δ within this sphere (with Lagrangian radius q_0) will be less than Δ . So, the sphere with Lagrangian radius, q_0 , would have expanded by a factor larger than $(1 + \Delta)^{-1/3}$ its present size being then larger than r .

Thus, the sphere with present radius r would have expanded from an initial one with $q' < q_0$ (neglecting shell crossing) and, assuming that the profile is monotonically increasing (which is obviously true for the most probable profile, but is also so for any realization with non-negligible probability), $\delta_l(q')$ shall be less than $\delta_l(q)$, and consequently, $\delta(r)$ shall be less than Δ . We may then write:

$$P_c(\Delta) \equiv P(\delta \leq \Delta, /r, \delta_1) = P(\delta_l \leq DL(\Delta), /q_0 = r(1 + \Delta)^{1/3}, \delta_1) \quad (\text{B1})$$

$$P(\delta/r, \delta_1) = \frac{d}{d\Delta} P_c(\Delta) \Big|_{\Delta=\delta} \quad (\text{B2})$$

where (B1) denote the fact that the probability that δ be smaller than Δ at r is, by the above arguments, equal to the probability that δ_l be smaller than $DL(\Delta)$ at q_0 . But for fixed q_0 and with the condition $\delta(R) = \delta_0$, δ_l follows distribution (39), so, expression (43) follows immediately.

We have assumed above that shell crossing does not occur, however this may be shown to be the case in the relevant applications. Note that the relevant shell crossing here is that on the scale of the void, that is the shell crossing experienced by the mass field filtered on the scale of the void. On much smaller scales there is obviously shell crossing, but this is irrelevant to our argument.