Mixed properties of MHD waves in non-uniform plasmas

M. Goossens [1,*], **I. Arregui** [2,3] **and T. Van Doorsselaere** [1]

[1] *Centre for mathematical Plasma Astrophysics, Department of Mathematics, KU Leuven, Celestijnenlaan 200B bus 2400, B-3001 Leuven, Belgium*
[2] *Instituto de Astrofísica de Canarias, E-38205 La Laguna, Tenerife, Spain*
[3] *Departamento de Astrofísica, Universidad de La Laguna, E-38206, La Laguna, Tenerife, Spain*

Correspondence*:
M. Goossens
marcel.goossens@kuleuven.be

ABSTRACT

This paper investigates the mixed properties of MHD waves in a non-uniform plasma. It starts with a short revision of MHD waves in a uniform plasma of infinite extent. In that case the MHD waves do not have mixed properties. They can be separated in Alfvén waves and magneto-sonic waves. The Alfvén waves propagate parallel vorticity and are incompressible. In addition they have no parallel displacement component. The magneto-sonic waves are compressible and in general do have a parallel component of displacement but do not propagate parallel vorticity. This clear separation has been the reason why there has been a strong inclination in the literature to use this classification in the study of MHD waves in non-uniform plasmas. The main part of this paper is concerned with MHD waves in a non-uniform plasma. It is shown that the MHD waves in that situation in general propagate both vorticity and compression and hence have mixed properties. Finally, the close connection between resonant absorption and MHD waves with mixed properties is discussed.

Keywords: magnetohydrodynamics (MHD), Sun: atmosphere, Sun: magnetic fields, Sun: corona, Sun: oscillations, waves

1 INTRODUCTION

Most textbooks on Magnetohydrodynamics (MHD) and plasma physics contain at least an elementary discussion of MHD waves in a uniform plasma of infinite extent (see e.g., Thompson, 1964; Mestel and Weiss, 1974; Goedbloed, 1983; Goossens, 2003; Walker, 2004; Goedbloed and Poedts, 2004). It is shown that the MHD waves are either Alfvén waves or slow/fast magneto-sonic waves. The Alfvén waves are incompressible and propagate parallel vorticity. They do not have a parallel component of displacement and are driven by magnetic tension only. The magneto-sonic waves are compressible and have a parallel component of displacement. They do not propagate parallel vorticity and are driven by pressure and magnetic tension. In non-uniform plasmas the situation can be very different. The clear division between Alfvén waves and magneto-sonic waves is no longer present. The MHD waves have mixed properties in non-uniform plasmas. Mixed properties mean that the general rule is that MHD waves propagate both parallel vorticity as in classic Alfvén waves and compression as in classic magneto-sonic waves. This behaviour causes exciting wave physics. For instance, the phenomenon of MHD waves with mixed properties can lead to damping, with relevance in explaining the attenuation observed in coronal and prominence oscillations and discussed by e.g., Goossens et al. (2002a, 2011); Terradas et al. (2006); Arregui et al. (2008); Pascoe et al. (2010, 2011), among many others. The use of the information on wave damping has also been found useful to perform solar coronal seismology (see e.g., Goossens et al., 2002a; Arregui et al., 2007; Goossens et al., 2008; Goossens, 2008). The mixed properties arise because in an inhomogeneous plasma the Eulerian perturbation of total pressure couples with the dynamics of the motion (Hasegawa and Uberoi, 1982). Mathematically this is translated into the fact that the differential equations for the radial component of the Lagrangian displacement ξ_r and the Eulerian perturbation of total pressure P' are coupled to algebraic equations for compression $\nabla \cdot \vec{\xi}$, the parallel and perpendicular projections of the Lagrangian displacement $\xi_{\parallel}, \xi_{\perp}$, and vorticity $\nabla \times \vec{\xi}$ [1]. The coupling of the equations is due to the coupling functions C_A and C_S which were introduced by Sakurai et al. (1991a) in their study of resonant absorption. The relevance of the coupling functions goes beyond resonant absorption. The spatial behaviour of the coupling functions and of the local Alfvén frequency ω_A and local cusp frequency ω_C determine the spatial behaviour of the various components of velocity and vorticity and of compression. The simultaneous presence of compression and vorticity is hard to avoid.

Goossens et al. (2009) investigated the forces that drive these waves and found that the magnetic tension force always dominates the pressure force for the kink mode. In addition, they showed that compression is small in the particular case of thin tubes. Hence, these waves do not have the typical properties of fast magneto-sonic waves and behave more as Alfvén waves. Goossens et al. (2011) reconsidered these waves in their section on quasi-modes and decided to call them surface Alfvén waves. In the present paper, we continue the theoretical investigation of the nature of the waves. In section 2, we describe pure Alfvén and pure magneto-acoustic waves in a uniform plasma of infinite extent, by analysing their eigenfrequencies, eigenfunctions, vorticity and compression. In section 3, the analysis is generalized to MHD waves in non-uniform plasmas, which propagate both compression and parallel vorticity at the same time. This leads to new expressions for the components of vorticity that are derived for axi-symmetric/non-axi-symmetric motions in a non-uniform 1-dimensional cylindrical plasma. In section 4, we show that resonant Alfvén /slow waves are characterized by strong shear in the perpendicular/parallel component of displacement with large values of the parallel/perpendicular component of vorticity. This strong shear causes violent KH-instabilities (Terradas et al., 2008; Antolin et al., 2018) that accelerate the damping of the MHD waves and facilitate heating of plasma (Antolin et al., 2015; Arregui, 2015; Terradas and Arregui, 2018).

2 LINEAR MHD WAVES OF A UNIFORM PLASMA OF INFINITE EXTENT

The properties of MHD waves in a uniform plasma of infinite extent are often used to characterize MHD waves in general. For a uniform plasma of infinite extent the MHD waves can be subdivided into two classes with distinct properties. The first class contains the magneto-sonic waves. They are compressive but do not propagate parallel vorticity. The second class contains the Alfvén waves. Alfvén waves propagate parallel vorticity and are incompressible. The equilibrium quantities are constant. The constant magnetic field

$$\vec{B_0} = B_0 \vec{1}_z, \tag{1}$$

[1] The standard definition of vorticity in fluid dynamics is $\nabla \times \vec{v}$. Here the analysis uses the Lagrangian displacement $\vec{\xi}$ and $\nabla \times \vec{\xi}$ is referred to as vorticity. Since $\vec{v} = -i\omega\vec{\xi}$ it follows that $\nabla \times \vec{v} = -i\omega \nabla \times \vec{\xi}$. $\nabla \cdot \vec{v}$ is a measure for the rate of variation of the volume of a material fluid element. In the present paper $\nabla \cdot \vec{\xi}$ is referred to as compression.

is used to define the direction of the z- axis of a Cartesian system of coordinates. The equilibrium density and pressure are constant

$$p_0 = \text{constant}, \;\; \rho_0 = \text{constant}. \tag{2}$$

In what follows $\vec{\xi}$ is the Lagrangian displacement. In the present subsection the background is static and uniform. As a consequence solutions can be obtained in the form of plane harmonic waves and $\vec{\xi}$ is written

$$\vec{\xi}(\vec{r};t) = \hat{\vec{\xi}} \exp(i(\vec{k}.\vec{r} - \omega t)) = \hat{\vec{\xi}} \exp(i(k_x x + k_y y + k_z z - \omega t)). \tag{3}$$

Here $\hat{\vec{\xi}}$ is the constant amplitude of $\vec{\xi}$, $\vec{k} = k_x \vec{1}_x + k_y \vec{1}_y + k_z \vec{1}_z$ is the wave vector, and ω is the frequency of the wave. In what follows the hat on $\vec{\xi}$ will be dropped. Since the constant magnetic field defines a preferred direction a clever choice of dependent wave variables is X, Y, Z defined as

$$\begin{aligned} k_z \xi_z = X &= \text{displacement parallel to } \vec{B}_0, \\ \nabla \cdot \vec{\xi} = i\, \vec{k} \cdot \vec{\xi} = i\, Y &= \text{compression}, \\ (\nabla \times \vec{\xi})_z = i\, (\vec{k} \times \vec{\xi})_z = i\, Z &= \text{component of vorticity parallel to } \vec{B}_0. \end{aligned} \tag{4}$$

X, Y, Z are dimensionless quantities and allow us to obtain an elegant version of the governing equations. In terms of these variables the equations for linear ideal MHD waves can be written as

$$\begin{aligned} \omega^2 X - k_z^2 v_S^2 Y &= 0, \\ k^2 v_A^2 X + (\omega^2 - k^2 (v_A^2 + v_S^2)) Y &= 0, \\ (\omega^2 - \omega_A^2) Z &= 0. \end{aligned} \tag{5}$$

v_A, v_S are the Alfvén velocity and the velocity of sound. They are defined by

$$v_A^2 = \frac{B_0^2}{\mu\, \rho_0}, \;\; v_S^2 = \frac{\gamma p_0}{\rho_0}. \tag{6}$$

ω_A is the local Alfvén frequency. It is defined as

$$\omega_A^2 = \frac{(\vec{k} \cdot \vec{B})^2}{\mu \rho} \;=\; k_z^2\, v_A^2 = k_\parallel^2\, v_A^2. \tag{7}$$

In a uniform plasma v_A, v_S, ω_A are constant. In a non-uniform plasma these quantities depend on position.

The system (5) consists of two uncoupled subsets of equations. The first subset is the third equation for the variable Z. The second subset contains the wave variables ξ_z and Y. The first type of MHD waves are characterized by

$$Y = 0, \;Z \neq 0, \;\xi_z = 0, \;\; \omega^2 = \omega_A^2. \tag{8}$$

They are the classic Alfvén waves. The eigenfrequencies associated with the Alfvén waves (8) are infinitely degenerate as they only depend on the parallel component of the wave vector $\vec{k}$. Alfvén waves do not cause compression and have no component of the displacement parallel to the magnetic field. They are the only waves that propagate parallel vorticity in a uniform plasma of infinite extent. The only restoring force is the magnetic tension force. Note also that Alfvén waves in a uniform plasma of infinite extent exist for any wave vector $\vec{k} = (k_x, k_y, k_z)$.

The displacement $\vec{\xi}$ for Alfvén waves is

$$\vec{\xi}_A = (-\frac{k_y}{k_x}\vec{1}_x + \vec{1}_y)\xi_y \; = \; (\vec{1}_x - \frac{k_x}{k_y}\vec{1}_y)\xi_x. \tag{9}$$

For $k_y = 0$ we obtain the popular result $\vec{\xi}_A = \xi_y \vec{1}_y$. These y- independent Alfvén waves are a special case. In the cylindrical case $k_y = 0$ and $k_y \neq 0$ correspond to respectively axisymmetric waves with $m = 0$ and to non-axisymmetric waves with $m \neq 0$ with m the azimuthal wave number. For a wave vector with both horizontal components of the wave vector different from zero both horizontal components of the displacement vector are non-zero. Let us now keep $k_y \neq 0, \;\; k_z \neq 0$ and mimic a situation with non-uniformity in the x- direction and a resonant condition where $\lim k_x \rightarrow +\infty$ so that $\mid k_y \mid \ll \mid k_x \mid, \mid k_z \mid \ll \mid k_x \mid$. Find then

$$\frac{\mid \xi_y \mid}{\mid \xi_x \mid} = \frac{\mid k_x \mid}{\mid k_y \mid} \gg 1, \;\; \vec{\xi}_A \approx \xi_y \; \vec{1}_y. \tag{10}$$

The motion in the Alfvén wave is predominantly in the y- direction and rapidly varying in the x- direction. The displacement (10) is not y- independent because of the factor $\exp(ik_y y)$ with $k_y \neq 0$. The $\approx$ sign means that the two components (ξ_x, ξ_y) are non-zero but ξ_y is far larger in absolute value than ξ_x. The two components are needed to satisfy the incompressibility condition.

For a general wave vector $\vec{k} = (k_x, k_y, k_z)^t$ the three components of vorticity $\nabla \times \vec{\xi}$ are non-zero. In addition to the parallel component $(\nabla \times \vec{\xi})_z$ also the components in planes normal to $\vec{B}_0$ are non-zero:

$$(\nabla \times \vec{\xi})_z = i(k_x \xi_y - k_y \xi_x), \;\; (\nabla \times \vec{\xi})_x = -ik_z \xi_y, \;\; (\nabla \times \vec{\xi})_y = ik_z \xi_x, \;\; \xi_x = -\frac{k_y}{k_x}\xi_y. \tag{11}$$

For our later discussion on resonant Alfvén waves it is instructive to look at the components of vorticity $\nabla \times \vec{\xi}$ under conditions that mimic resonant behaviour, i.e. when $\mid k_y \mid \ll \mid k_x \mid, \mid k_z \mid \ll \mid k_x \mid$ and find that

$$\frac{\mid (\nabla \times \vec{\xi})_z \mid}{\mid (\nabla \times \vec{\xi})_x \mid} \approx \frac{\mid k_x \mid}{\mid k_z \mid} \gg 1, \;\; \frac{\mid (\nabla \times \vec{\xi})_z \mid}{\mid (\nabla \times \vec{\xi})_y \mid} \approx \frac{\mid k_x \mid}{\mid k_z \mid}\frac{\mid k_x \mid}{\mid k_y \mid} \gg 1.$$

Hence

$$\mid (\nabla \times \vec{\xi})_y \mid \ll \mid (\nabla \times \vec{\xi})_x \mid \ll \mid (\nabla \times \vec{\xi})_z \mid,$$

so that

$$\nabla \times \vec{\xi} \approx (\nabla \times \vec{\xi})_z \; \vec{1}_z \approx ik_x \; \xi_y \; \vec{1}_z. \tag{12}$$

Here also the $\approx$ sign means that the three components $(\nabla \times \vec{\xi})$ are non-zero but the parallel component is far larger in absolute value than the two horizontal components.

The second class of MHD waves corresponds to

$$Y \neq 0, \;\; Z = 0, \;\; \xi_z = \xi_{\parallel} \neq 0. \tag{13}$$

They are the magneto-sonic waves. They cause compression but do not propagate parallel vorticity. However, they cause horizontal vorticity. Their displacement has a component parallel to the magnetic field that is driven by the magnetic pressure force. The dispersion relation is

$$(\omega^2)^2 - k^2(v_S^2 + v_A^2)\omega^2 + k_z^2 k^2 v_S^2 v_A^2 = 0. \tag{14}$$

The well-known solutions for the eigenfrequencies are

$$\omega^2 = \omega_{sl,f}^2 = \frac{k^2(v_S^2 + v_A^2)}{2}\left\{1 \pm \left(1 - \frac{4\omega_C^2}{k^2(v_S^2 + v_A^2)}\right)^{1/2}\right\}. \tag{15}$$

$k^2 = k_x^2 + k_y^2 + k_z^2$, ω_C and v_C are the cusp frequency, and the cusp velocity.

$$\omega_C^2 = \frac{v_S^2}{v_S^2 + v_A^2}\omega_A^2 = k_{||}^2 v_C^2 = k_z^2 v_C^2, \;\; v_C^2 = \frac{v_S^2 v_A^2}{v_S^2 + v_A^2}. \tag{16}$$

In equation (15) "sl" corresponds to the minus sign, and "f" to the plus sign. The corresponding waves are the slow and fast magneto-sonic waves. The frequencies of the magneto-sonic waves depend on the three components (k_x, k_y, k_z) of the wave vector $\vec{k}$. They depend in the same way on k_x and k_y because of isotropy in the planes normal to $\vec{B}_0$. It is instructive to consider the variation of $\omega_{sl,f}^2$ as function of k_x for fixed values of (k_y, k_z). The cut-off frequencies ω_I, ω_{II} are defined as

$$\omega_I^2 = \omega_{sl}^2(k_x = 0, k_y, k_z), \;\; \omega_{II}^2 = \omega_f^2(k_x = 0, k_y, k_z). \tag{17}$$

Also

$$\lim_{k_x \to \infty} \omega_{sl}^2 = \omega_C^2, \;\; \lim_{k_x \to \infty} \omega_f^2 = \infty \tag{18}$$

The cut-off frequencies $\omega_I, \; \omega_{II}$ and the characteristic frequencies $\omega_A, \; \omega_C$ obey the sequence of inequalities

$$\omega_C^2 \leq \omega_{sl}^2 \leq \omega_I^2 \leq \omega_A^2 \leq \omega_{II}^2 \leq \omega_f^2 < +\infty. \tag{19}$$

Hence the spectrum of linear motions of a uniform plasma of infinite extent can be divided in a slow subspectrum $]\omega_C, \; \omega_I]$, a degenerate Alfvén point spectrum ω_A and a fast subspectrum $[\omega_{II}, \; +\infty[$. The first equality in (18) means that ω_C is an accumulation point of the slow subspectrum.

The magneto-sonic waves are driven by tension and pressure forces and cause variations in density and pressure and horizontal vorticity.

The solutions for the eigenfunctions are

$$\vec{\xi}_{sl,f} = (\vec{1}_x + \frac{k_y}{k_x}\;\vec{1}_y + \frac{\omega_{sl,f}^2 - k^2 v_A^2}{\omega_{sl,f}^2}\;\frac{k_z}{k_x}\vec{1}_z)\;\xi_x, \tag{20}$$

or equivalently,

$$\vec{\xi}_{sl,f} = (\frac{\omega_{sl,f}^2}{\omega_{sl,f}^2 - k^2 v_A^2}\;\frac{k_x}{k_z}\vec{1}_x + \frac{\omega_{sl,f}^2}{\omega_{sl,f}^2 - k^2 v_A^2}\;\frac{k_y}{k_z}\vec{1}_y + \vec{1}_z)\xi_z. \tag{21}$$

The popular view is that the horizontal motion (ξ_x, ξ_y) is the dominant motion for fast waves while the parallel motion ξ_z is the dominant motion for slow waves. In order to point out that this is not the general rule, ξ_x is used as the measuring unit in (20) and ξ_z in (21). It is straightforward to show that in general the parallel component in (20) is not small compared to the horizontal components, and similarly that the horizontal components in (21) are not per se much smaller than the parallel component. However, for strong magnetic fields, i.e. $v_A \gg v_S$ it can be shown that

$$\vec{\xi}_f \approx (\vec{1}_x + \frac{k_y}{k_x}\, \vec{1}_y)\, \xi_x; \;\; \vec{\xi}_{sl} \approx \xi_z \vec{1}_z. \tag{22}$$

The popular view corresponds to the limiting case of a strong field.

The parallel component of vorticity $(\nabla \times \vec{\xi})_z = iZ$ is of course zero. However the horizontal components are non-zero

$$\nabla \times \vec{\xi} = -i\, k_z\, \frac{k^2\, v_A^2}{\omega_{sl,f}^2}\, \xi_x\, (\frac{k_y}{k_x}\vec{1}_x - \vec{1}_y). \tag{23}$$

For $k_y = 0$ the expressions (20) for the displacement $\vec{\xi}$ and (23) for vorticity $\nabla \times \vec{\xi}$ can be simplified to

$$\vec{\xi}_{sl,f} = (\vec{1}_x + \frac{\omega_{sl,f}^2 - k^2 v_A^2}{\omega_{sl,f}^2}\, \frac{k_z}{k_x}\vec{1}_z)\, \xi_x, \;\; \nabla \times \vec{\xi} = i\, k_z\, \frac{k^2\, v_A^2}{\omega_{sl,f}^2}\, \xi_x\, \vec{1}_y. \tag{24}$$

Keep $k_y \neq 0,\; k_z \neq 0$ and finite and mimic a situation with non-uniformity in the x- direction and a turning point where $k_x = 0$ and find

$$\vec{\xi}_{sl,f} = (\frac{\omega_{I,II}^2}{\omega_{I,II}^2 - k^2 v_A^2}\, \frac{k_y}{k_z}\vec{1}_y + \vec{1}_z)\xi_z. \tag{25}$$

In summary for a uniform plasma of infinite extent the division is clear. The equation for vorticity is uncoupled from the equations for compression and parallel displacement. The waves have either parallel vorticity and no compression and no parallel displacement, these are the Alfvén waves, or they have compression and parallel displacement and no parallel vorticity, they are magneto-sonic waves. There are no waves with compression and parallel vorticity at the same time. There is no mixing of properties.

For a pressureless plasma with

$$v_S^2 = 0, \tag{26}$$

the solutions for the magnetosonic waves are

$$\begin{aligned} \omega_C^2 &= 0, \;\; \omega_{sl}^2 = 0, \;\; \omega_f^2 = k^2 v_A^2, \\ \xi_z &= 0, \;\; \vec{\xi}_f = (\vec{1}_x + \frac{k_y}{k_x}\, \vec{1}_y)\, \xi_x, \;\;\; \nabla \times \vec{\xi} = -i\, k_z\;\, \xi_x\, (\frac{k_y}{k_x}\vec{1}_x - \vec{1}_y). \end{aligned} \tag{27}$$

In this situation there are no slow waves and the fast magneto-sonic waves have no parallel motions. The parallel motions are driven by the gradient of plasma pressure and here plasma pressure vanishes by assumption. The absence of slow waves and of parallel motions is a general result for a pressureless plasma.

In what follows, no particular attention will be devoted to pressureless plasmas. The equations for MHD waves for a pressureless plasma are easily obtained by putting $v_S^2 = 0$ in the general equations.

3 MIXED PROPERTIES IN NON-UNIFORM PLASMAS

The aim of the present section is to show that MHD waves in a non-uniform plasma have mixed properties. In general they propagate compression and parallel vorticity at the same time. The phenomenon of mixed properties follows from the fact that the equations that describe the linear motions are coupled, unlike for the case of a uniform plasma of infinite extent. In particular the focus is on MHD waves on 1-D cylindrical plasma columns. The equilibrium model is a straight cylindrical plasma column of radius R in static equilibrium. In what follows we use cylindrical coordinates r, φ, z. The magnetic field has both an axial and an azimuthal component

$$\vec{B_0} = B_{z,0}\vec{1}_z + B_{\varphi,0}\vec{1}_\varphi. \tag{28}$$

The equilibrium density $\rho_0(r)$, equilibrium pressure $p_0(r)$ and the components of the equilibrium magnetic field $B_{z,0}(r), B_{\varphi,0}(r)$ are functions of r or constant. The equilibrium quantities satisfy the equation of static equilibrium

$$\frac{d}{dr}(p_0 + \frac{B_0^2}{2\mu}) = -\frac{B_{\varphi,0}^2}{\mu r}, \;\; B_0^2 = B_{\varphi,0}^2 + B_{z,0}^2. \tag{29}$$

In a nonuniform plasma v_S^2, v_A^2, ω_A^2, and ω_C^2 are functions of position. In what follows f' and δf denote respectively the Eulerian and Lagrangian variation of a quantity f. In linear theory they are related as

$$\delta f = f' + \frac{df_0}{dr}\xi_r, \tag{30}$$

with f_0 the equilibrium value of f. In the following equations $P' = p' + \vec{B_0} \cdot \vec{B'}/\mu$ is the Eulerian perturbation of total pressure; p' is the Eulerian perturbation of plasma pressure. $\vec{\xi}$ is the Lagrangian displacement.

We use the mixed field line / magnetic surface triad $(\vec{b}, \vec{n}, \vec{\pi})$ defined by Goedbloed et al. (2010) in their Equations 17.23. In the present case of a straight cylindrical flux tube with the equilibrium magnetic field $\vec{B_0}$ defined in the equation (28)

$$\vec{n} = \vec{1}_r, \; \vec{b} = \vec{1}_B = \vec{1}_\|, \; \vec{\pi} = \vec{1}_\perp,$$

with $\vec{1}_\|, \vec{1}_\perp$ the unit vectors in the magnetic surfaces respectively parallel and perpendicular to the magnetic field lines.

ξ_r is the radial component of Lagrangian displacement and $\xi_\|, \xi_\perp$ are the projections of the Lagrangian displacement in the magnetic surfaces parallel and perpendicular to the magnetic field lines:

$$\xi_\perp = (\xi_\varphi B_{z,0} - \xi_z B_{\varphi,0})/B, \;\; \xi_\| = \vec{\xi} \cdot \vec{B_0}/B_0. \tag{31}$$

Since the equilibrium quantities are independent of φ and z the wave variables can be put proportional to the exponential factor $\exp(i(m\varphi + k_z z))$ with m, k_z the azimuthal and axial wave numbers, m is an integer. For example, for the Lagrangian displacement we write

$$\vec{\xi}(\vec{r}; t) = \vec{\hat{\xi}}(r) \exp(i(m\varphi + k_z z - \omega t)). \tag{32}$$

$\hat{\vec{\xi}}(r)$ is the radially varying amplitude of $\vec{\xi}$. In what follows the hat on $\vec{\xi}$ and on the other wave variables will be omitted. It is convenient to introduce the wave vector $\vec{k} = (0, m/r, k_z)$.

The linear MHD waves can be described by two ordinary differential equations for ξ_r and P' (see e.g., Appert et al. 1974; Sakurai et al. 1991a; Goossens et al. 1992, 1995)

$$\begin{aligned} D\frac{d(r\ \xi_r)}{dr} &= C_1 r\ \xi_r - C_2 r\ P', \\ D\frac{dP'}{dr} &= C_3 \xi_r - C_1 P'. \end{aligned} \tag{33}$$

The coefficient functions D, C_1, C_2, C_3 are given by

$$\begin{aligned} D &= \rho_0 (v_S^2 + v_A^2)(\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2), \\ C_1 &= \frac{2}{\mu r} B_{\varphi,0}^2\, \omega^4 - (v_S^2 + V_A^2)(\omega^2 - \omega_C^2)\frac{2 m f_B}{\mu r^2} B_{\varphi,0}, \\ C_2 &= \omega^4 \ - (v_S^2 + v_A^2)(\omega^2 - \omega_C^2)\left(\frac{m^2}{r^2} + k_z^2\right) = (\omega^2 - \omega_I^2)(\omega^2 - \omega_{II}^2), \\ C_3 &= D\left[\rho_0\ (\omega^2 - \omega_A^2) + \frac{2B_{\varphi,0}}{\mu}\frac{d}{dr}\left(\frac{B_{\varphi,0}}{r}\right)\right] \\ &+ \frac{4\omega^4 B_{\varphi,0}^4}{\mu^2 r^2} - 4\rho_0\ (v_S^2 + v_A^2)(\omega^2 - \omega_C^2)\omega_A^2 \frac{B_{\varphi,0}^2}{\mu r^2}. \end{aligned} \tag{34}$$

v_A and v_S are the Alfvén speed and the speed of sound as before and are defined in equation (6). In a non-uniform plasma they are functions of position. The quantities f_B and g_B are defined as

$$f_B = \vec{k} \cdot \vec{B_0} = k_z B_{z,0} + \frac{m\ B_{\varphi,0}}{r}, \quad g_B = (\vec{k} \times \vec{B_0})\vec{1}_r = \frac{m\ B_{z,0}}{r} - k_z B_{\varphi,0}. \tag{35}$$

The frequencies ω_A and ω_C are the local Alfvén frequency and the local cusp frequency as before. They are defined for the planar case in equation (7). Here in the cylindrical case their squares are defined as

$$\omega_A^2 = \frac{f_B^2}{\mu\rho_0} = \frac{\left(k_z B_{z,0} + \dfrac{m}{r} B_{\varphi,0}\right)^2}{\mu\rho_0}, \quad \omega_C^2 = \frac{v_S^2}{v_S^2 + v_A^2}\omega_A^2. \tag{36}$$

Note that ω_A and ω_C are functions of position. For a given set of wave numbers (m, k_z) ω_A and ω_C map out two ranges of frequencies known as the Alfvén continuum and the cusp continuum. The frequencies $\omega_I,\ \omega_{II}$ are defined as

$$\omega_{I,II}^2 = \frac{1}{2}(\frac{m^2}{r^2} + k_z^2)\ (v_S^2 + v_A^2)\left\{1 \pm \left[1 - \frac{4\omega_C^2}{(\dfrac{m^2}{r^2} + k_z^2)(v_S^2 + v_A^2)}\right]^{1/2}\right\}. \tag{37}$$

They are the cylindrical analogues of the Cartesian cut-off frequencies defined in (17). Here they are not cut-off frequencies but rather frequencies that restrict the Sturmian or anti-Sturmian behaviour of the spectrum as explained by Goedbloed (1975, 1983).

To emphasize that parallel motions are solely driven by the gradient plasma pressure force, the parallel component of the equation of motion is written as

$$\rho_0\, \omega^2\, \xi_{\|} = \frac{i f_B}{B_0}\, \delta p. \tag{38}$$

δp is the Lagrangian variation of plasma pressure.

For the discussion of the mixed properties it is necessary to look at the wave variables $\xi_{\perp},\ \xi_{\|}, \nabla \cdot \vec{\xi}$ and $(\nabla \times \vec{\xi})$. They are given by expressions in ξ_r and P' and their derivatives. Algebraic expressions for $\xi_{\perp},\ \xi_{\|}, \nabla \cdot \vec{\xi}$ can be found in e.g. Sakurai et al. (1991a)

$$\begin{aligned} \rho_0(\omega^2 - \omega_A^2)\xi_{\perp} &= \frac{i}{B_0}\, C_A, \\ \rho_0(\omega^2 - \omega_C^2)\xi_{\|} &= \frac{i f_B}{B_0}\, \frac{v_S^2}{v_S^2 + v_A^2}\, C_S, \\ \nabla \cdot \vec{\xi} &= \frac{-\omega^2}{\rho_0\, (v_S^2 + v_A^2)(\omega^2 - \omega_C^2)}\, C_S. \end{aligned} \tag{39}$$

The coupling functions are defined as (see e.g., Sakurai et al., 1991a)

$$C_A = g_B\; P' - \frac{2 f_B B_{\varphi,0} B_{z,0}}{\mu r}\, \xi_r, \quad C_S = P' - \frac{2 B_{\varphi,0}^2}{\mu r}\, \xi_r. \tag{40}$$

They are linear combinations of P' and ξ_r. The coefficients of ξ_r in C_A and C_S vanish when the equilibrium magnetic field is straight $B_{\varphi,0} = 0$. C_A depends on the azimuthal wave number m and the longitudinal wave number k_z. C_S on the other hand is independent of the wave numbers (m, k_z). The coupling functions play an essential role for the mixing properties of MHD waves and for resonant absorption. They are called coupling functions for the good reason that they couple the differential equations (33) for ξ_r and P' to the expressions for all of the remaining wave variables $\xi_{\perp},\ \xi_{\|}, \nabla \cdot \vec{\xi},\ (\nabla \times \vec{\xi})$. First they couple the differential equations for ξ_r and P' to the algebraic equations (39) for $\xi_{\perp},\ \xi_{\|}, \nabla \cdot \vec{\xi}$. When $C_A \neq 0$ the first equation of (39) implies that $\xi_{\perp} \neq 0$. Similarly when $C_S \neq 0$ the second and third equation of (39) imply that $\nabla \cdot \vec{\xi} \neq 0$. When in addition to $C_S \neq 0$ also $v_S \neq 0$ it follows that $\xi_{\|} \neq 0$.

Let us now consider $(\nabla \times \vec{\xi})$. In section 2 it was pointed out that a division of linear waves can be based on compression, parallel displacement and parallel vorticity. A characterization based on the components (ξ_x, ξ_y, ξ_z) is in general not possible. When we move from Cartesian geometry to cylindrical geometry the horizontal components (ξ_x, ξ_y) are replaced by the components $(\xi_r, \xi_{\perp})$ in the planes normal to $\vec{B}_0$ and ξ_z is replaced by the component $\xi_{\|}$ parallel to the equilibrium magnetic field. For a uniform plasma of infinite extent, the MHD waves could be divided into incompressible waves that propagate parallel vorticity, i.e. the Alfvén waves and waves that propagate compression and have a parallel displacement component i.e. the magneto-sonic waves. In what follows it will be shown that for a non-uniform plasma MHD waves propagate both compression and parallel vorticity and have non-zero radial, perpendicular and parallel components of displacement and vorticity. To the best of our knowledge expressions for the components of $(\nabla \times \vec{\xi})$ are not available in the literature. They are

$$
\begin{aligned}
(\nabla \times \xi)_r &= i\,\frac{g_B}{B_0}\,\xi_\parallel - i\,\frac{f_B}{B_0}\,\xi_\perp, \\
(\nabla \times \vec{\xi})_\parallel &= \frac{d\xi_\perp}{dr} + P_\perp \xi_\perp + P_\parallel \xi_\parallel - i\frac{g_B}{B_0}\xi_r, \\
(\nabla \times \vec{\xi})_\perp &= -\frac{d\xi_\parallel}{dr} + Q_\perp \xi_\perp + Q_\parallel \xi_\parallel + i\frac{f_B}{B_0}\xi_r.
\end{aligned} \tag{41}
$$

Expressions for $P_\perp, P_\parallel, Q_\perp, Q_\parallel$ are

$$
\begin{aligned}
P_\perp &= \frac{B_{z,0}}{B_0}\frac{1}{r}\frac{d}{dr}\left(\frac{rB_{z,0}}{B_0}\right) + \frac{B_{\varphi,0}}{B_0}\frac{d}{dr}\left(\frac{B_{\varphi,0}}{B_0}\right), \\
P_\parallel &= \frac{B_{z,0}}{B_0}\frac{1}{r}\frac{d}{dr}\left(\frac{rB_{\varphi,0}}{B_0}\right) - \frac{B_{\varphi,0}}{B_0}\frac{d}{dr}\left(\frac{B_{z,0}}{B_0}\right), \\
Q_\perp &= \frac{B_{z,0}}{B_0}\frac{d}{dr}\left(\frac{B_{\varphi,0}}{B_0}\right) - \frac{B_{\varphi,0}}{B_0}\frac{1}{r}\frac{d}{dr}\left(\frac{rB_{z,0}}{B_0}\right), \\
Q_\parallel &= -\frac{B_{z,0}}{B_0}\frac{d}{dr}\left(\frac{B_{z,0}}{B_0}\right) + \frac{B_{\varphi,0}}{B_0}\frac{1}{r}\frac{d}{dr}\left(\frac{rB_{\varphi,0}}{B_0}\right).
\end{aligned}
$$

The equations (41) show that the components of $(\nabla \times \vec{\xi})$ can be expressed in terms of $(\xi_r, \xi_\perp, \xi_\parallel)$. Since $\xi_\perp, \xi_\parallel$ are expressed in terms of ξ_r and P' it follows that also the components of $(\nabla \times \vec{\xi})$ can be expressed in terms of ξ_r and P'. When $(\xi_r, \xi_\perp, \xi_\parallel)$ are non-zero, the components of vorticity are in general also non-zero. All of the wave variables are coupled. The MHD waves have mixed properties, they propagate both compression and parallel vorticity and have non-zero radial, perpendicular and parallel components of displacement and vorticity. In general all wave variables are non-zero. A situation in which a subset of the wave variables is not coupled to the other wave variables is an exception. Such a situation will appear for axi-symmetric motions in the presence of a straight field. The clear division into Alfvén waves and magneto-sonic waves that exists for a uniform plasma of infinite extent does not any longer hold.

Hence in general for linear MHD waves on a non-uniform plasma

$$
\begin{aligned}
\xi_r &\neq 0, \quad P' \neq 0, \\
\xi_\perp &\neq 0, \quad \xi_\parallel \neq 0, \\
\nabla \cdot \vec{\xi} &\neq 0, \quad (\nabla \times \vec{\xi}) \neq 0.
\end{aligned} \tag{42}
$$

Let us consider the special case of axi-symmetric motions with $m = 0$. The expressions for $f_B,\ g_B,\ C_A,\ C_S$ can be simplified to

$$\begin{aligned} f_B &= k_z B_{z,0}, \quad g_B = -k_z\ B_{\varphi,0}, \\ C_A &= -k_z B_{\varphi,0}\{P' + 2\ \frac{B_{z,0}^2\ \xi_r}{\mu r}\}, \quad C_S = P' - 2\frac{B_{\varphi,0}^2}{\mu r}\xi_r. \end{aligned} \tag{43}$$

For a twisted magnetic field with both a longitudinal component $B_{z,0}$ and a non-zero azimuthal component $B_{\varphi,0}$, the coupling functions C_A and C_S are non-zero. This implies that the preceding analysis on mixed properties also applies to axi-symmetric motions. The axi-symmetric motions propagate vorticity and compression. The situation is different when the magnetic field is straight.

Since C_A and C_S are functions of position the coupling of the equations depends on position and so does the strength of the mixing of the wave properties. For example a wave can start off as a predominantly fast wave, change into a wave that has both fast and Alfvén properties and turn into a predominantly Alfvénic wave. MHD waves have mixed properties and have different appearances in different parts of the plasma because of the inhomogeneity of the plasma. This phenomenon was discussed by e.g., Goossens et al. (2002b); Goossens (2008); Goossens et al. (2011, 2012, 2014). Waves with mixed properties are also referred to as coupled MHD waves (Pascoe et al., 2010, 2011). This is a rather strange name as it seems to suggest that there are two or more waves involved.

Let us now focus on MHD waves in the presence of a straight field. For a straight field ($B_{\varphi,0} = 0$) the magnetic surfaces are cylinders: $r =$ constant, and the φ- and z- directions are the directions in the magnetic surfaces respectively perpendicular and parallel to the magnetic field lines. The r- direction is normal to the magnetic surfaces. Hence ξ_r is associated with motions normal or across magnetic surfaces; $\xi_\parallel = \xi_z$ are motions along the magnetic field lines and $\xi_\perp = \xi_\varphi$ are motions in the magnetic surfaces perpendicular to the magnetic field lines. For a straight field the expressions for $f_B,\ g_B,\ C_A,\ C_S$ are simplified to

$$\begin{aligned} f_B &= k_z B_{z,0}, \quad g_B = \frac{m}{r} B_{z,0}, \\ C_A &= g_B\ P' = \frac{m}{r} B_{z,0}\ P', \quad C_S = P'. \end{aligned} \tag{44}$$

The coupling functions $C_A,\ C_S$ only contain P'. The coefficients of ξ_r in C_A and C_S vanish when $B_{\varphi,0} = 0$. Hence the coupling of the waves variables is solely due to P' as will become clear in what follows. As far as the wave numbers (m, k_z) are concerned, C_A no longer depends on k_z, only on m.

The differential equations (33) for ξ_r and P' and the algebraic equations for $\xi_\perp,\ \xi_\parallel, \nabla \cdot \vec{\xi}$ (39) are now

$$\begin{aligned}
D\frac{d(r\xi_r)}{dr} &= -C_2 r P', \\
\frac{dP'}{dr} &= \rho_0\,(\omega^2-\omega_A^2)\xi_r, \\
\rho_0\,(\omega^2-\omega_A^2)\xi_\varphi &= \frac{im}{r}P', \\
\rho_0(\omega^2-\omega_C^2)\xi_z &= ik_z\frac{v_S^2}{v_S^2+v_A^2}P', \quad \rho_0\,\omega^2\xi_z = ik_z\delta p, \\
\nabla\cdot\vec{\xi} &= \frac{-\omega^2 P'}{\rho_0\,(v_S^2+v_A^2)(\omega^2-\omega_C^2)}.
\end{aligned} \tag{45}$$

Use now (41) and note that for a straight field

$$P_\perp = \frac{1}{r}, \;\; P_\| = 0, \;\; Q_\perp = 0, \;\; Q_\| = 0,$$

to obtain

$$\begin{aligned}
(\nabla\times\xi)_r &= i(\frac{m}{r}\xi_z - k_z\xi_\varphi), \\
(\nabla\times\xi)_\varphi &= -\frac{d\xi_z}{dr} + ik_z\xi_r, \\
(\nabla\times\xi)_z &= \frac{d\xi_\varphi}{dr} + \frac{\xi_\varphi}{r} - i\frac{m}{r}\xi_r.
\end{aligned} \tag{46}$$

Equations (45) and (46) govern the MHD waves on a non-uniform straight cylindrical plasma column with a straight magnetic field. There is a natural subdivision between respectively axi-symmetric motions with $m = 0$ and non-axisymmetric motions with $m \neq 0$. The reason being that the equation for ξ_φ for $m = 0$ is decoupled from the remaining equations. Let us first focus on axi-symmetric motions with $m = 0$.

$$C_A = 0, \;\; C_S = P'. \tag{47}$$

The equation for $\xi_\perp = \xi_\varphi$ is decoupled from the remaining equations

$$\rho_0\,(\omega^2-\omega_A^2)\,\xi_\varphi = 0. \tag{48}$$

Equation (48) can be satisfied in two ways. First of all by choosing

$$\omega^2 = \omega_A^2, \;\; \xi_\varphi \neq 0. \tag{49}$$

The second choice is

$$\omega^2 \neq \omega_A^2, \;\; \xi_\varphi = 0. \tag{50}$$

The solutions given in (49) and (50) correspond respectively to the axi-symmetric Alfvén waves and the sausage magneto-sonic waves. The axi-symmetric MHD waves are decoupled in sausage magneto-sonic waves and axi-symmetric Alfvén waves. The solutions for the axi-symmetric magneto-sonic waves are

$$\begin{aligned}
& P' \neq 0, \\
& \xi_r \neq 0, \; \xi_z \neq 0, \; \xi_\varphi = 0, \\
& \nabla \cdot \vec{\xi} = \frac{-\omega^2 P'}{\rho_0 (v_S^2 + v_A^2)(\omega^2 - \omega_C^2)} \neq 0, \\
& (\nabla \times \vec{\xi})_r = 0, \; (\nabla \times \vec{\xi})_z = 0. \\
& (\nabla \times \vec{\xi})_\varphi = -ik_z \frac{d}{dr} \left\{ \frac{v_S^2}{v_A^2 + v_S^2} \frac{1}{\rho_0(\omega^2 - \omega_C^2)} \right\} P' \\
+ \;\; & ik_z \frac{\omega^2}{\rho_0 (\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2)} \frac{v_A^2}{v_A^2 + v_S^2} \frac{dP'}{dr}.
\end{aligned} \tag{51}$$

The solutions for the axi-symmetric Alfvén waves are

$$\begin{aligned}
& P' = 0, \\
& \xi_r = 0, \;\; \xi_z = 0, \;\; \xi_\varphi \neq 0, \\
& \nabla \cdot \vec{\xi} = 0, \\
& (\nabla \times \vec{\xi})_r = -ik_z \xi_\varphi, \;\; (\nabla \times \vec{\xi})_\varphi = 0, \;\; (\nabla \times \vec{\xi})_z = \frac{1}{r}\frac{d}{dr}(r\xi_\varphi).
\end{aligned} \tag{52}$$

For an axi-symmetric non-uniform 1-dimensional cylindrical plasma this is the only case where pure Alfvén waves show up in the analysis. Each magnetic surface oscillates with its own local Alfvén frequency. In a twisted magnetic field, $C_A \neq 0$ for $m = 0$ so that the equations are coupled and the corresponding MHD waves have mixed magneto-acoustic and Alfvén properties. Also $C_S \neq 0$ for any azimuthal wave number m. The absence of pure Alfvén waves in a non-uniform 1-D cylindrical plasma for azimuthal wave numbers $m \neq 0$ is in stark contrast to the situation for a magnetic flux tube with piece wise constant density and magnetic field. Spruit (1982) showed that solutions with $\nabla \cdot \vec{v} = 0$ exist for any m. H. Spruit correctly identified these solutions as Alfvén waves. Flow patterns for Alfvén waves with $m = 0$ and $m = 1$ are shown on Figure 1 in Spruit (1982). In addition to the Alfvén waves there are compressive waves. The fact that pure non-axisymmetric Alfvén waves do not exist in a non-uniform straight plasma cylinder is an illustration of how the non-uniformity produces waves with mixed properties.

Let us now turn back to the non-axisymmetric MHD waves with $m \neq 0$. Actually the analysis also holds for axi-symmetric MHD waves with $\xi_\varphi = 0$. Excluded from the analysis are the axi-symmetric Alfvén waves defined in (52). The equation (46) can be rewritten as

$$\begin{aligned}
(\nabla \times \xi)_r &= k_z \frac{m}{r} \frac{v_A^2}{v_S^2 + v_A^2} \frac{\omega^2}{\rho_0(\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2)} P', \\
(\nabla \times \xi)_\varphi &= -ik_z \frac{d}{dr} \left\{ \frac{v_S^2}{v_A^2 + v_S^2} \frac{1}{\rho_0(\omega^2 - \omega_C^2)} \right\} P' \\
&+ ik_z \frac{\omega^2}{\rho_0(\omega^2 - \omega_A^2)(\omega^2 - \omega_C^2)} \frac{v_A^2}{v_A^2 + v_S^2} \frac{dP'}{dr}, \\
(\nabla \times \xi)_z &= -i\frac{m}{r} \frac{1}{\left\{\rho_0(\omega^2 - \omega_A^2)\right\}^2} \frac{d}{dr} \left\{\rho_0(\omega^2 - \omega_A^2)\right\} P'
\end{aligned} \tag{53}$$

Note that the expressions for the components of vorticity for axi-symmetric magneto-sonic waves can be obtained from (53) by putting $m = 0$.

Here all wave variables are coupled and all wave variables are non-zero. In case of a straight field, it is the non-zero Eulerian perturbation of total pressure $P' \neq 0$ that produces MHD waves with mixed properties reminiscent of Alfvén waves and magneto-sonic waves. See also the comments by Hasegawa and Uberoi (1982) in their Chapter 3 on MHD waves in an inhomogeneous medium.

Special interest goes to the components of $\nabla \times \xi$. It is obvious that $(\nabla \times \xi)_r \neq 0$ irrespective if the equilibrium is uniform or not. The same is true for $(\nabla \times \xi)_\varphi$. The second term is always non-zero. The first term is non-zero for a non-uniform equilibrium and for a piece-wise constant density model the derivative results in a delta-function contribution. The parallel component $(\nabla \times \xi)_z$ is non-zero for a non-uniform equilibrium with

$$\frac{d}{dr} \left\{\rho_0(\omega^2 - \omega_A^2)\right\} \tag{54}$$

different from zero. In a fully non-uniform equilibrium this condition is satisfied everywhere. In a piece-wise constant density model the derivative results in a delta-function contribution.

Let us try to understand the cause of the vorticity. The equilibrium model is a 1-D straight cylinder with the equilibrium quantities functions of the radial distance r to the axis. There is no baroclinic source of vorticity since the iso-surfaces of density and pressure coincide. Equations (41) combined with the expressions for $P_\perp, P_\parallel, Q_\perp, Q_\parallel$ in principle contain all the information. They are rather complicated and do not allow a straightforward interpretation. Physical insight can be gained by considering the case of a straight field. For a straight field the equation of motion in the horizontal planes follows from the 2nd and 3rd equations of Equation (45).

$$-\rho_0\, \omega^2\, \vec{\xi}_h = -\nabla_h P' - \rho_0\, \omega_A^2\, \vec{\xi}_h. \tag{55}$$

$\vec{\xi}_h$ is the displacement in horizontal planes and ∇_h is the gradient operator in horizonal planes

$$\vec{\xi}_h = (\xi_r, \xi_\varphi, 0), \quad \nabla_h = (\frac{d}{dr},\ i\frac{m}{r},\ 0).$$

The left hand side of Equation (55) is mass density times acceleration. The first term in the right hand side of Equation (55) is the horizontal gradient total pressure force; the second term is mass density times the magnetic tension force

$$\vec{T} = -\omega_A^2 \ \vec{\xi_h}, \quad -\frac{1}{\rho_0}\nabla_h P' = -(\omega^2 - \omega_A^2)\vec{\xi_h}.$$

Hence

$$-\frac{1}{\rho_0}\nabla_h P' = \frac{\omega^2 - \omega_A^2}{\omega_A^2} \ \vec{T}. \tag{56}$$

The importance of tension force compared to the horizontal pressure force depends on the frequency of the wave. When $\omega^2 \approx \omega_A^2$ the magnetic tension force dominates; when $\omega^2 >> \omega_A^2$ then the the horizontal pressure force dominates; when $\omega^2 << \omega_A^2$ the horizontal pressure force and the magnetic tension force are of equal strength. For other values of ω^2 the actual ratio has to be computed.

From Equation (55)

$$\omega^2 \ (\nabla \times \vec{\xi_h}) = \nabla \times (\frac{1}{\rho_0}\nabla_h P') - \nabla \times \vec{T}. \tag{57}$$

This shows that vorticity generated by the horizontal motions is due to the horizontal component of the gradient pressure force and the magnetic tension force. Equation (55) can be solved for ξ_h as

$$\vec{\xi_h} = \Phi \ \nabla_h P', \quad \Phi = \frac{1}{\rho_0(\omega^2 - \omega_A^2)}. \tag{58}$$

We can use equation (56) to estimate for the relative contribution of the magnetic tension force and the horizontal gradient pressure force to the vorticity. Since $(\omega^2 - \omega_A^2)/\omega_A^2$ is non-constant in a non-uniform plasma we anticipate that the magnetic tension force is the dominant contributor to vorticity for $\omega^2 \approx \omega_A^2$; while the horizontal pressure force is the dominant contributor for $\omega^2 >> \omega_A^2$. Since

$$\nabla \times \nabla_h P' = k_z \frac{m}{r} P' \vec{1}_r + i k_z \frac{dP'}{dr} \vec{1}_\varphi,$$

the result for vorticity is

$$\nabla \times \vec{\xi_h} = \frac{d\Phi}{dr} \ \frac{im}{r} \ P' \vec{1}_z + k_z \ \Phi \ \left\{ \frac{m}{r} P' \ \vec{1}_r + i \frac{dP'}{dr} \ \vec{1}_\varphi \right\} \ . \tag{59}$$

Equation (59) follows from Equation (53) when we remove from this equation the contribution due to the parallel motions.

In the same manner, we can consider the equation of motion parallel to the magnetic field lines. From the 4th equation of Equation (45) it follows that ξ_z is given by

$$\xi_z = i k_z \ \Psi \ P', \quad \Psi = \frac{1}{\rho_0(\omega^2 - \omega_C^2)} \ \frac{v_S^2}{v_S^2 + v_A^2}. \tag{60}$$

The result for vorticity associated with the parallel motion is then

$$\nabla \times (\xi_z \vec{1}_z) = i k_z \left\{ -\frac{d\Psi}{dr} P' \vec{1}_\varphi + \Psi(-\frac{dP'}{dr}\vec{1}_\varphi + \frac{i\,m}{r} P' \vec{1}_r) \right\}. \quad (61)$$

This shows that vorticity generated by the parallel motions is due to the gradient pressure and vanishes in a pressureless plasma when $v_S^2 = 0$. The sum of $\nabla \times \vec{\xi}_h$ given by Equation (59) and $\nabla \times (\xi_z \vec{1}_z)$ given by Equation (61) is equal to the result given in Equation (53).

Equations (58) and (59) show that the horizontal motions and vorticity associated with horizontal motions are controlled by the function

$$\Phi = \frac{1}{\rho_0(\omega^2 - \omega_A^2)}.$$

Conversely Equations (60) and (61) show that the parallel motions and vorticity associated with parallel motions are controlled by the function

$$\Psi = \frac{1}{\rho_0(\omega^2 - \omega_C^2)} \frac{v_S^2}{v_S^2 + v_A^2}.$$

For non-axisymmetric MHD waves on a non-uniform plasma column with a straight magnetic field all wave variables are non-zero and coupled. The coupling factor is P'. This means that any given variable can be expressed in terms of another wave variable. Let us see what we can do with for example compression and parallel vorticity. Together with the parallel displacement ξ_z these are the two quantities that were used in section 2 to distinguish between Alfvén waves and magneto-sonic waves. The expressions for compression $\nabla \cdot \vec{\xi}$ and for parallel vorticity $(\nabla \times \vec{\xi})_z$ for non-axisymmetric motions in a straight field can be rewritten in compact form as

$$\nabla \cdot \vec{\xi} = N_C\; P', \quad (\nabla \times \vec{\xi})_z = i\; m\; N_V\; P', \quad (62)$$

with

$$N_C \;=\; \frac{-\omega^2}{\rho_0\;(v_S^2 + v_A^2)(\omega^2 - \omega_C^2)}, \quad N_V = \frac{-1}{r\left\{\rho_0(\omega^2 - \omega_A^2)\right\}^2} \frac{d}{dr} \left\{\rho_0(\omega^2 - \omega_A^2)\right\}. \quad (63)$$

The ratio of parallel vorticity to compression is

$$\frac{\mid (\nabla \times \vec{\xi})_z \mid}{\mid \nabla \cdot \vec{\xi} \mid} = \mid m \mid \; \frac{\mid N_V \mid}{\mid N_C \mid}. \quad (64)$$

In addition to the parallel component of vorticity also the components in horizontal planes, i.e. $(\nabla \times \vec{\xi})_\varphi$, $(\nabla \times \vec{\xi})_r$ are as a rule non-zero in a non-uniform plasma. MHD waves turn out to be very efficient in situ generators of vorticity in non-uniform plasmas. This equation shows that a non-axisymmetric compressional motion immediately generates vorticity and vice versa a vortical motion generates compression. It is impossible to have one property without the other one. MHD waves that propagate compression but no vorticity or vice versa do not exist. The waves have always mixed properties.

The cylindrical model with a straight magnetic field has a Cartesian analogue. The Cartesian version has a vertical magnetic field along the z- axis and the direction of inhomogeneity along the x- axis. The cylindrical case with a straight field and axi-symmetric waves with $m = 0$ then corresponds to $k_y = 0$. For $k_y = 0$ the Cartesian equations for the wave variables are decoupled in equations for the magneto-sonic waves and equations for Alfvén waves. However, for $k_y \neq 0$ the equations are coupled and the MHD waves have mixed properties. Examples of this behaviour can be found in e.g., Tirry and Berghmans (1997); Tirry et al. (1997); De Groof and Goossens (2000, 2002); De Groof et al. (2002).

4 RESONANT ABSORPTION OF MHD WAVES

Let us turn to the discussion of resonant absorption and resonant MHD waves. We have already pointed out that the coupling functions C_A and C_S depend on position. This implies that the strength of the mixing of the wave properties depends on position. MHD waves have mixed properties and have different appearances in different parts of the plasma because of the inhomogeneity of the plasma. The phenomenon that the properties of MHD waves change as the wave propagates through a non-uniform environment is most clearly at work in resonant absorption. For example, in case of resonant Alfvén waves the MHD wave arrives at a position where it can behave as an almost pure Alfvén wave. Similarly, in case of resonant cusp waves the MHD arrives at a position where it can behave as a slow wave for perpendicular propagation. Resonant absorption and resonant waves have been discussed previously, see e.g., Goossens et al. (2011). We shall review aspects related to the displacement components $\xi_r,\ \xi_\perp,\ \xi_\parallel$ and P'. We shall focus on the behaviour of compression $\nabla \cdot \vec{\xi}$ and vorticity $\nabla \times \vec{\xi}$ for resonant waves. The coupling functions C_A and C_S play an important role here also. Look back at the expression for the coefficient function D (34). The local Alfvén frequency $\omega_A(r)$ and the local cusp frequency $\omega_C(r)$ vary with position r and they map out two intervals of frequencies

$$\text{AC} = [\min\ \omega_A(r),\ \ \max\ \omega_A(r)\ ],\quad \text{SC} = [\min\ \omega_C(r),\ \ \max\ \omega_C(r)\ ]$$

They are known as the Alfvén continuum (AC) and the slow or cusp continuum (SC) (Appert et al., 1974; Chen and Hasegawa, 1974; Goedbloed, 1983). For a frequency ω either in the Alfvén continuum or the slow continuum the coefficient function $D = 0$ at the position r_A where the frequency is equal to the local Alfvén frequency $\omega = \omega_A(r_A)$ or at the position r_C where the frequency is equal to the local cusp frequency $\omega = \omega_C(r_C)$. The system of differential equations (33) have regular singular points at the positions $r = r_A$ and $r = r_C$.

Let us first consider the Alfvén continuum. For a frequency in the Alfvén continuum the dispersion relation for Alfvén waves is locally satisfied. Each magnetic surface oscillates at its own Alfvén continuum frequency. Let us determine the structure of the MHD wave with a frequency in the Alfvén continuum. The MHD waves live on $[0, +\infty[$ in the r- direction. Solutions over the full spatial interval can be found in e.g. Poedts et al. (1989, 1990); Sakurai et al. (1991b); Goossens and Poedts (1992); Tirry and Goossens (1996); Ruderman and Roberts (2002); Van Doorsselaere et al. (2004); Soler et al. (2013). Away from the resonant surface the MHD wave can be predominantly magneto-sonic. During its propagation through the non-uniform plasma the MHD wave might change in a wave that has both magneto-sonic and Alfvén properties. Close to and at the resonant surface the MHD wave is almost completely an Alfvén wave. Here we focus on the spatial behaviour close to the singular point $r = r_A$ where $\omega = \omega_A(r_A)$. We follow Sakurai et al. (1991a); Goossens et al. (1992, 1995); Tirry and Goossens (1996). They used Frobenius-Fuchs solutions around the singular point $r = r_A$ where $\omega = \omega_A(r_A)$ and introduced a new radial variable

$s = r - r_A$. This analysis is valid in the interval $[-s_A, s_A]$ where the linear Taylor polynomial is an accurate approximation of $\omega^2 - \omega_A^2(r)$:

$$\omega^2 - \omega_A^2 \approx \Delta_A \, s, \quad \Delta_A = \frac{d}{dr}(\omega^2 - \omega_A^2)_{r_A} \tag{65}$$

The outcome of the application of the Frobenius-Fuchs method is the fundamental conservation law for resonant Alfvén waves

$$C_A(s) \equiv g_B P' - \frac{2 f_B B_{\varphi,0} \, B_{z,0}}{\mu r_A} \, \xi_r = \text{constant}, \tag{66}$$

and the solutions for ξ_r and P'

$$\begin{aligned} \xi_r(s) &= \frac{g_B}{\rho_0 \, B_0^2 \, \Delta_A} C_A \ln(\mid s \mid) + \left\{ \begin{array}{ll} \xi_- & s < 0 \\ \xi_+ & s > 0, \end{array} \right. \\ P'(s) &= \frac{2 f_B B_{\varphi,0} B_{z,0}}{\mu r_A \, \rho_0 \, B_0^2 \Delta_A} C_A \ln(\mid s \mid) + \left\{ \begin{array}{ll} P'_- & s < 0 \\ P'_+ & s > 0. \end{array} \right. \end{aligned} \tag{67}$$

All equilibrium quantities are evaluated at $s = 0$ ($r = r_A$). The solutions for ξ_r and P' are characterized by a logarithmic singularity and a jump. The jump in a quantity Q is defined as

$$[Q] = \lim_{s \to 0^+} Q(s) - \lim_{s \to 0^-} Q(s).$$

Recall the equation for $\xi_\perp$

$$\rho_0 (\omega^2 - \omega_A^2) \xi_\perp = \frac{i}{B_0} \, C_A.$$

Hence close to $s = 0$

$$s \, \xi_\perp \approx i \, \frac{C_A}{\rho_0 \, B_0 \, \Delta_A}. \tag{68}$$

This means that $\xi_\perp$ has a $1/s$- singularity and a $\delta(s)$- contribution. These singularities dominate those present in ξ_r and P'. The equation for the parallel component $\xi_\parallel$ is

$$\rho_0 (\omega^2 - \omega_C^2) \xi_\parallel = \frac{i f_B}{B_0} \, \frac{v_S^2}{v_S^2 + v_A^2} \, C_S.$$

The coefficient of $\xi_\parallel$ in the left hand side of this equation is finite and non-zero for frequencies in the Alfvén continuum. The function C_S is a linear combination of ξ_r and P' and can contain a logarithmic term $\ln(\mid s \mid)$ and a jump. This implies that $\xi_\parallel$ contains at most a logarithmic term $\ln(\mid s \mid)$ and a jump and is dominated by $\xi_\perp$. Hence close to $s = 0$ we are in a situation that closely resembles that described in equation (10) when we make the transformation

$$x \to r, \quad y \to \perp, \quad z \to \parallel \tag{69}$$

and note that

$$\mid \xi_\parallel \mid \leq \mid \xi_r \mid \ll \mid \xi_\perp, \quad \vec{\xi}_A \approx \xi_\perp \vec{1}_\perp. \tag{70}$$

The motion (70) in the Alfvén wave is predominantly in the $\perp$- direction and rapidly varying in the r-direction. The $\approx$ sign means that the three components $(\xi_{\parallel}, \xi_r, \xi_{\perp})$ are non-zero but $\xi_{\perp}$ is far larger in absolute value than the two other components.

Consider now equations (41) for the components of $\nabla \times \vec{\xi}$ and identify the first term in the right hand side of the equation for $(\nabla \times \vec{\xi})_{\parallel}$ as the dominant term overall. Hence

$$
\begin{aligned}
(\nabla \times \vec{\xi})_{\parallel} &\approx \frac{d\xi_{\perp}}{dr} = \frac{d}{dr}\left\{\frac{1}{\rho_0(\omega^2 - \omega_A^2)}\frac{i\, C_A}{B_0}\right\} \\
&\approx -\frac{i\, C_A}{B_0}\frac{1}{\left\{\rho_0(\omega^2 - \omega_A^2)\right\}^2}\frac{d}{dr}\left\{\rho_0(\omega^2 - \omega_A^2)\right\} \\
&\approx -i\frac{C_A}{\rho_0\, B_0\, \Delta_A}\,\frac{1}{s^2}. \qquad (71)
\end{aligned}
$$

$(\nabla \times \vec{\xi})_{\parallel}$ has a $1/s^2$- singularity. The remaining components $(\nabla \times \vec{\xi})_r$ and $(\nabla \times \vec{\xi})_{\perp}$ are non-zero and both have a $1/s$- singularity when $f_B \neq 0$ and $Q_{\perp} \neq 0$. Hence use the transformation (69) to help us to identify the inequalities (12) but now as

$$
\mid (\nabla \times \vec{\xi})_{\perp} \mid \leq \mid (\nabla \times \vec{\xi})_r) \mid \ll \mid (\nabla \times \vec{\xi})_{\parallel} \mid, \qquad (72)
$$

so that

$$
\nabla \times \vec{\xi} \approx (\nabla \times \vec{\xi})_{\parallel}\vec{1}_{\parallel}. \qquad (73)
$$

Here also the $\approx$ sign means that the three components $(\nabla \times \vec{\xi})$ are non-zero but the parallel component is far larger in absolute value than the two horizontal components.

In summary

$$
\lim_{s\to 0}\frac{\mid \xi_{\perp} \mid}{\mid \xi_r \mid} = +\infty, \quad \lim_{s\to 0}\frac{\mid \xi_{\perp} \mid}{\mid \xi_{\parallel} \mid} = +\infty, \quad \lim_{s\to 0}\frac{\mid (\nabla \times \vec{\xi})_{\parallel} \mid}{\mid (\nabla \times \vec{\xi})_{r,\perp} \mid} = +\infty. \qquad (74)
$$

Hence the dominant dynamics is in the perpendicular motions. The jumps in ξ_r and P' (67) are due to dissipative effects. At and in the vicinity of the resonant position $s = 0$ the resonant MHD wave has very strong Alfvén wave properties. Its properties match the properties derived on the basis of very simple principles for Alfvén waves that mimic a resonant situation in section 2. The resonant Alfvén wave is linked to the outside world by the coupling function C_A.

A comment on the case of axi-symmetric motions with $m = 0$. The expressions for $f_B,\ g_B,\ C_A,\ C_S$ for axi-symmetric motions are given in (43). In particular it was pointed out that for a twisted magnetic field with both a longitudinal component $B_{z,0}$ and a non-zero azimuthal component $B_{\varphi,0}$ the coupling functions C_A and C_S are non-zero. Hence the preceding analysis on resonant properties also applies to axi-symmetric motions. Resonant absorption of axi-symmetric motions in the Alfvén continuum was investigated by Giagkiozis et al. (2016) for an a non-straight magnetic field and in the slow continuum by Yu et al. (2017a,b) for a straight magnetic field. In addition, the preceding analysis for the behaviour of the various variables can be repeated for a pressureless plasma. The additional simplification is that

$\xi_{\parallel} = 0$. The behaviour of the resonant waves at and in the vicinity of the resonant position is to a large extent insensitive to plasma pressure.

The mathematical results in (68) for $\xi_{\perp}$ and (71) for $(\nabla \times \vec{\xi})_{\parallel}$ mean that there are strong counterstreaming flows in the perpendicular direction at and close to $s = 0$. Of course in reality infinite values for $\xi_{\perp}$ do not occur. We shall see that these infinite values are replaced by finite and very large values. This is the basis for the Kelvin-Helmholtz instability in Alfvén waves first investigated by Terradas et al. (2008) and subsequently studied by several groups (see e.g., Antolin et al., 2014, 2015, 2018).

Let us now turn to the slow continuum. The analysis for a frequency in the slow continuum parallels that for Alfvén waves (see e.g., Sakurai et al., 1991a; Goossens and Ruderman, 1995). The MHD waves live on $[0, \infty[$. Here we focus on the spatial behaviour close to the singular point $r = r_C$ where $\omega = \omega_C(r_C)$. The variable s is now defined as $s = r - r_C$ with r_C the position where $\omega^2 = \omega_C^2(r_C)$. This analysis is valid in the interval $[-s_C, s_C]$ where the linear Taylor polynomial is an accurate approximation of $\omega^2 - \omega_C^2(r)$:

$$\omega^2 - \omega_C^2 \approx \Delta_C \; s, \quad \Delta_C = \frac{d}{dr}(\omega^2 - \omega_C^2)_{r_C}. \tag{75}$$

The outcome is the fundamental conservation law for resonant slow waves

$$C_S(s) \equiv P' - \frac{2B_{\varphi,0}^2 \; \xi_r}{\mu r} = \text{constant}, \tag{76}$$

and the solutions for ξ_r and P'

$$\begin{aligned} \xi_r(s) &= \frac{\omega_C^4}{(B_0^2/\mu)\; \omega_A^2 \; \Delta_C} C_S \; \ln(|\; s \;|) + \begin{cases} \xi_- & s < 0 \\ \xi_+ & s > 0, \end{cases} \\ P'(s) &= \frac{2\; \omega_C^4 \; B_{\varphi,0}^2}{r_C \; B_0^2 \; \omega_A^2 \; \Delta_C} C_S \ln(|\; s \;|) + \begin{cases} P'_- & s < 0 \\ P'_+ & s > 0. \end{cases} \end{aligned} \tag{77}$$

Recall the equation for $\xi_{\parallel}$

$$\rho_0(\omega^2 - \omega_C^2)\xi_{\parallel} = \frac{i f_B}{B_0} \; \frac{v_S^2}{v_S^2 + v_A^2} \; C_S.$$

Hence close to $s = 0$

$$s \; \xi_{\parallel} = \frac{i f_B}{B_0 \; \rho_0 \; \Delta_C} \; \frac{v_S^2}{v_S^2 + v_A^2} \; C_S. \tag{78}$$

This means that $\xi_{\parallel}$ has a $1/s$- singularity and a $\delta(s)$- contribution. These singularities dominate those present in ξ_r and P'.

The equation for the perpendicular component $\xi_{\perp}$ is

$$\rho_0(\omega^2 - \omega_A^2)\xi_{\perp} = \frac{i}{B_0} \; C_A.$$

The coefficient of $\xi_\perp$ in the left hand side of this equation is finite and non-zero for frequencies in the cusp continuum. The function C_A is a linear combination of ξ_r and P' and can contain a logarithmic term $\ln(\mid s \mid)$ and a jump. This implies that $\xi_\perp$ contains at most a logarithmic term $\ln(\mid s \mid)$ and a jump and is dominated by $\xi_\parallel$. Hence close to $s = 0$ we are in a situation

$$\mid \xi_\perp \mid \leq \mid \xi_r \mid \ll \mid \xi_\parallel \mid, \quad \vec{\xi_S} \approx \xi_\parallel \vec{1}_\parallel. \tag{79}$$

The motion (79) in the slow wave is predominantly in the $\|$- direction and rapidly varying in the r- direction. The $\approx$ sign means that the three components $(\xi_\parallel, \xi_r, \xi_\perp)$ are non-zero but $\xi_\parallel$ is far larger in absolute value than the two other components.

Recall from (39) the equation for $\nabla \cdot \vec{\xi}$ as

$$\nabla \cdot \vec{\xi} = \frac{-\omega^2}{\rho_0\ (v_S^2 + v_A^2)(\omega^2 - \omega_C^2)}\ C_S$$

and find that in the vicinity of $s = 0$ $\nabla \cdot \vec{\xi}$ behaves as

$$s\ (\nabla \cdot \vec{\xi}) = -\frac{\omega_C^2}{\rho_0\ \Delta_C\ (v_S^2 + v_A^2)}\ C_S. \tag{80}$$

$\nabla \cdot \vec{\xi}$ has a $1/s$- singularity and a $\delta(s)$- contribution in the same way as $\xi_\parallel$.

Consider now equations (41) for the components of $\nabla \times \vec{\xi}$ and identify the first term in the right hand side of the equation for $(\nabla \times \vec{\xi})_\perp$ as the dominant term overall. Hence

$$\begin{aligned}
(\nabla \times \vec{\xi})_\perp &\approx -\frac{d\xi_\parallel}{dr} = -\frac{d}{dr}\left\{\frac{1}{\rho_0(\omega^2 - \omega_C^2)}\frac{if_B}{B_0}\ \frac{v_S^2}{v_S^2 + v_A^2}\ C_S\right\} \\
&\approx \frac{if_B}{B_0}\ \frac{v_S^2}{v_S^2 + v_A^2}\ C_S\ \frac{1}{\left\{\rho_0(\omega^2 - \omega_C^2)\right\}^2}\frac{d}{dr}\left\{\rho_0(\omega^2 - \omega_C^2)\right\} \\
&\approx i\frac{f_B}{\rho_0\ B_0\ \Delta_C}\ \frac{v_S^2}{v_S^2 + v_A^2}\ C_S\ \frac{1}{s^2}.
\end{aligned} \tag{81}$$

$(\nabla \times \vec{\xi})_\perp$ has a $1/s^2$-singularity. The remaining components $(\nabla \times \vec{\xi})_r$ and $(\nabla \times \vec{\xi})_\parallel$ are non-zero and both have a $1/s$- singularity when $g_B \neq 0$ and $P_\parallel \neq 0$.

$$\mid (\nabla \times \vec{\xi})_\parallel \mid \leq\ \mid (\nabla \times \vec{\xi})_r) \mid\ \ll\ \mid (\nabla \times \vec{\xi})_\perp \mid, \tag{82}$$

so that

$$\nabla \times \vec{\xi}\ \ \approx\ \ (\nabla \times \vec{\xi})_\perp \vec{1}_\perp. \tag{83}$$

Here also the $\approx$ sign means that the three components $(\nabla \times \vec{\xi})$ are non-zero but the perpendicular component is far larger in absolute value than the radial and parallel components.

In summary

$$\lim_{s\to 0}\frac{\mid \xi_{\|} \mid}{\mid \xi_r \mid} = +\infty, \quad \lim_{s\to 0}\frac{\mid \xi_{\|} \mid}{\mid \xi_{\perp} \mid} = +\infty, \quad \lim_{s\to 0}\frac{\mid (\nabla\times\vec{\xi})_{\perp} \mid}{\mid (\nabla\times\vec{\xi})_{r,\|} \mid} = +\infty. \tag{84}$$

The dominant dynamics is in the component in the magnetic surfaces and parallel to the magnetic field lines. In the vicinity of the resonant magnetic surface the wave is almost exactly a slow wave in a homogeneous plasma with its wave vector almost perpendicular to the magnetic field.

The mathematical results in (78) for $\xi_{\|}$ and (81) for $\nabla\times\vec{\xi}_{\perp}$ mean that there are strong counterstreaming flows in the parallel direction at and close to $s = 0$. Of course in reality infinite values for $\xi_{\perp}$ do not occur. We shall see that these infinite values are replaced by finite and very large values. Our prediction is that this is the basis for the Kelvin-Helmholtz instability in slow resonant waves. This possible Kelvin-Helmholtz instability has not yet been studied.

5 CONCLUSIONS

Pure Alfvén waves and pure magneto-acoustic waves exist in a uniform plasma of infinite extent. In a non-uniform plasma the MHD waves combine the properties of the classic Alfvén waves and of magneto-sonic waves in a uniform plasma of infinite extent. The mixing of the properties is controlled by the coupling functions C_A and C_S. The general rule is that MHD waves in a non-uniform plasma propagate both compression and parallel vorticity and that the parallel, perpendicular and radial components of displacement and vorticity are non-zero. Vortex motions driven by MHD waves are as far as we can anticipate not different from vortex motions generated by other sources. Our analysis shows that MHD waves in non-uniform plasmas are very efficient in situ generators of vorticity. In a non-uniform plasma MHD waves can fill the whole space with vorticity. Vortex motions are expected to be very prominent where resonant conditions are satisfied. The signatures of vortex motions in the process of resonant Alfvén damping are very strong sheared azimuthal motions. Observational aspects of these strong sheared azimuthal motions and possible turbulent behaviour have been investigated by Okamoto et al. (2015) and compared to results of numerical simulations by Antolin et al. (2015). Of course in a pressureless plasma the parallel component of the displacement is zero. The exception to the general rule of mixed properties are axi-symmetric waves in the presence of a straight magnetic field. The coupling functions depend on position. Hence as an MHD waves propagates through a non-uniform plasma its properties change. Resonant absorption is a clear example of this phenomenon. In case of resonant Alfvén waves the MHD wave arrives at a position where it behaves as an almost pure Alfvén wave. Similarly, in case of resonant cusp waves the MHD arrives at a position where it behaves as a slow wave for perpendicular propagation. Resonant absorption for MHD waves with frequencies in the Alfvén / slow continuum is controlled by the coupling functions C_A and C_S. Analysis of the motions associated with the resonant Alfvén /slow waves shows that the resonant waves are characterized by strong shear in the perpendicular/parallel component of displacement with large values of the parallel/perpendicular component of vorticity. This strong shear causes violent KH-instabilities that accelerate the damping of the MHD waves and facilitate heating of plasma.

CONFLICT OF INTEREST STATEMENT

The authors declare that the research was conducted in the absence of any commercial or financial relationships that could be construed as a potential conflict of interest.

AUTHOR CONTRIBUTIONS

The three authors contributed in equal parts to this paper.

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